QUANTUM ENTANGLEMENT FOR SYSTEMS OF IDENTICAL BOSONS. II. SPIN SQUEEZING AND OTHER ENTANGLEMENT TESTS

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Abstract

These two accompanying papers are concerned with entanglement for systems of identical massive bosons and the relationship to spin squeezing and other quantum correlation effects. The main focus is on two mode entanglement, but multi-mode entanglement is also considered. The bosons may be atoms or molecules as in cold quantum gases. The previous paper I dealt with the general features of quantum entanglement and its specific definition in the case of systems of identical bosons. Entanglement is a property shared between two (or more) quantum sub-systems. In defining entanglement for systems of identical massive particles, it was concluded that the single particle states or modes are the most appropriate choice for sub-systems that are distinguishable, that the general quantum states must comply both with the symmetrisation principle and the super-selection rules (SSR) that forbid quantum superpositions of states with differing total particle number (global SSR compliance). Further, it was concluded that (in the separable states) quantum superpositions of sub-system states with differing sub-system particle number (local SSR compliance) also do not occur. The present paper II determines possible tests for entanglement based on the treatment of entanglement set out in paper I.

Several inequalities involving variances and mean values of operators have been previously proposed as tests for entanglement between two sub-systems. These inequalities generally involve mode annihilation and creation operators and include the inequalities that define spin squeezing. In this paper, spin squeezing criteria for two mode systems are examined, and spin squeezing is also considered for principle spin operator components where the covariance matrix is diagonal. The proof, which is based on our SSR compliant approach shows that the presence of spin squeezing in any one of the spin components requires entanglement of the relevant pair of modes. A simple Bloch vector test for entanglement is also derived. Thus we show that spin squeezing bdecomes a rigorous test for entanglement in a system of massive bosons, when viewed as a test for entanglement between two modes.

In addition, other previously proposed tests for entanglement involving spin operators are considered, including those based on the sum of the variances for two spin components. All of the tests are still valid when the present concept of entanglement based on the symmetrisation and super-selection rule criteria is applied. These tests also apply in cases of multi-mode entanglement, though with restrictions in the case of sub-systems each consisting of pairs of modes. Tests involving quantum correlation functions are also considered and for global SSR compliant states these are shown to be equivalent to tests involving spin operators. A new weak correlation test is derived for entanglement based on local SSR compliance for separable states, complementing the stronger correlation test obtained previously when this is ignored. The Bloch vector test is equivalent to one case of this weak correlation test. Quadrature squeezing for single modes is also examined but not found to yield a useful entanglement test, whereas two mode quadrature squeezing proves to be a valid entanglement test, though not as useful as the Bloch vector test. The various entanglement tests are considered for well-known entangled states, such as binomial states, rela-

tive phase eigenstates and NOON states - sometimes the new tests are satisfied whilst than those obtained in other papers are not.

The present paper II then outlines the theory for a simple two mode interferometer showing that such an interferometer can be used to measure the mean values and covariance matrix for the spin operators involved in entanglement tests for the two mode bosonic system. The treatment is also generalised to cover multi-mode interferometry. The interferometer involves a pulsed classical field characterised by a phase variable and an area variable defined by the time integral of the field amplitude, and leads to a coupling between the two modes. For simplicity the centre frequency was chosen to be resonant with the inter-mode transition frequency. Measuring the mean and variance of the population difference between the two modes for the output state of the interferometer for various choices of interferometer variables is shown to enable the mean values and covariance matrix for the spin operators for the input quantum state of the two mode system to be determined. The paper concludes with a discussion of several key experimental papers on spin squeezing.

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Contents

1 Introduction

The previous paper I dealt with the general features of quantum entanglement and its specific definition in the case of systems of identical bosons. In defining entanglement for systems of identical massive particles, it was concluded that the single particle states or modes are the most appropriate choice for sub-systems that are distinguishable. Further, it was conclude that the general quantum states must comply both with the symmetrisation principle and the super-selection rules (SSR) forbidding quantum superpositions of states with differing total particle number (global SSR compliance). As a consequence, it was then reasoned that in the separable states quantum superpositions of sub-system states with differing sub-system particle number (local SSR compliance) do not occur [1]. Other approaches - such as sub-systems consisting of labelled indistinguishable particles and entanglement due to symmetrisation [2] or allowing for non-entangled separable but non-local states [3]- were found to be unsuitable. The local (and global) SSR compliant definition of entanglement used here was justified on the basis of there being no non-relativistic quantum processes available to create SSR non-compliant states and alternatively on the absence of a phase reference [4].

Paper I can be summarised as follows. Section **2** covered the key definitions of entangled states, the relationship to hidden variable theory and some of the key paradoxes associated with quantum entanglement such as EPR and Bell inequalities. A detailed discussion on why the symmetrisation principle and the super-selection rule is invoked for entanglement in identical particle systems was discussed in Section **3**. Challenges to the necessity of the super-selection rule were outlined, with arguments against such challenges dealt with in Appendices **D** and **E**. Two key mathematical inequalities were derived in Appendix **B** and details about the spin EPR paradox set out in Appendix **C**. The final Section **4** summarised the key features of quantum entanglement discussed in the paper.

The present paper II focuses on tests for entanglement in two mode systems of identical bosons, with particular emphasis on spin squeezing and correlation tests and how the quantities involved in these tests can be measured via two mode interferometry. Two mode bosonic systems are of particular interest because cold atomic gases cooled well below the Bose-Einstein condensation (BEC) transition temperature can be prepared where essentially only two modes are occupied ([5], [6]). This can be achieved for cases involving a single hyperfine components using a double well trap potential or for two hyperfine components using a single well. At higher temperatures more than two modes may be occupied, so multi-mode systems are also of importance and the two mode treatment is extended to this situation.

As well as their relevance for entanglement tests, states that are spin squeezed have important applications in *quantum metrology*. That squeezed states can improve interferometry via the quantum noise in quadrature variables being reduced to below the standard quantum limit has been known since the pioneeering work of Caves [7] on optical systems. The extension to spin squeezing in systems of massive bosons originates with the work of Kitagawa and Ueda

[8], who considered systems of two state atoms. As this review is focused on spin squeezing as an entanglement test rather than the use of spin squeezing in quantum metrology, the latter subject will not be covered here. In quantum metrology involving spin operators the quantity $\sqrt{\left\langle \Delta\widehat{S}_x^2 \right\rangle}/|\left\langle \widehat{S}_z \right\rangle|$ (which involves the variance and mean value of orthogonal spin operators) is a measure of the uncertainty $\Delta\theta$ in measuring the interferometer *phase*. The interest in spin squeezing lies in the possibility of improvement over the *standard quantum limit* where $\Delta\theta = 1/\sqrt{N}$ (see Subsection 3.7). As we will see, for squeezed states $\sqrt{\left\langle \Delta\widehat{S}_x^2 \right\rangle} < \sqrt{|\left\langle \widehat{S}_z \right\rangle|/2}$ so we could have $\Delta\theta <\sim 1/\sqrt{|\left\langle \widehat{S}_z \right\rangle|} \sim 1/\sqrt{N}$ which is less than the standard quantum limit. In SubSection 3.8 we give an example of a highly squeezed state where $\Delta\theta \sim \sqrt{\ln N}/N$ which is near the *Heisenberg limit*. Suffice to say that increasing the number of particles in the squeezed state has the effect of improving the sensitivity of the interferometer. Aspects of quantum metrology are covered in a number of papers (see [9], [10]), based on concepts such as quantum Fisher information, Cramers-Rao bound [11], [12], quantum phase eigenstates.

The proof of the key conclusion that spin squeezing in any spin component is a sufficiency test for entanglement [1] is set out in this paper, as is that for a new Bloch vector test. A previous proof [13] that spin squeezing in the z spin component $\sqrt{\left\langle \Delta\widehat{S}_z^2 \right\rangle} < \sqrt{|\left\langle \widehat{S}_x \right\rangle|/2}$demonstrates entanglement based on treating identical bosonic atoms as distinguishable sub-systems has therefore now been superceded. It is seen that correlation tests for entanglement of quantum states complying with the global particle number SSR can be expressed in terms of inequalities involving powers of spin operators. Section 2 sets out the definitions of spin squeezing and in the following Section 3 it is shown that spin squeezing is a signature of entanglement, both for the original spin operators with entanglement of the original modes, for the principle spin operators with entanglement of the two new modes and finally for several multi-mode cases. Details of the latter are set out in Appendices 11 and 12. A number of other correlation, spin operator and quadrature operator tests for entanglement proposed by other authors are considered in Sections 4, 5 and 6, with details of these treatments set out in Appendices 14 and 15. Some tests also apply in cases of multi-mode entanglement, though with restrictions in the case of sub-systems each consisting of pairs of modes. A new weak correlation test is derived and for one case is equivalent to the Bloch vector test.

In Section 7 it is shown that a simple two mode interferometer can be used to measure the mean values and covariance matrix for the spin operators involved in entanglement tests, with details covered in Appendices 17 and 18. The treatment is also generalised to cover multi-mode interferometry. Actual experiments aimed at detecting entanglement via spin squeezing tests are examined in Section 8. The final Section 9 summarises and discusses the key results regarding entanglement tests. Appendices 16 and 19 provide details regarding certain important states whose features are discussed in this paper - the

"separable but non-local " states and the relative phase eigenstate.

2 Spin Squeezing

The basic concept of spin squeezing was first introduced by Kitagawa and Ueda [8] for general spin systems. These include cases based on two mode systems, such as may occur both for optical fields and for Bose-Einstein condensates. Though focused on systems of massive identical bosons, the treatment in this paper also applies to photons though details will differ.

2.1 Spin Operators, Bloch Vector and Covariance Matrix

2.1.1 Spin Operators

For two mode systems with mode annihilation operators $\widehat{a}$, $\widehat{b}$ associated with the two single particle states $|\phi_a\rangle$, $|\phi_b\rangle$, and where the non-zero bosonic commutation rules are $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$), Schwinger *spin angular momentum operators* $\widehat{S}_\xi$ ($\xi = x, y, z$) are defined as

$$\widehat{S}_x = (\widehat{b}^\dagger\widehat{a} + \widehat{a}^\dagger\widehat{b})/2 \qquad \widehat{S}_y = (\widehat{b}^\dagger\widehat{a} - \widehat{a}^\dagger\widehat{b})/2i \qquad \widehat{S}_z = (\widehat{b}^\dagger\widehat{b} - \widehat{a}^\dagger\widehat{a})/2 \tag{1}$$

and which satisfy the commutation rules $[\widehat{S}_\xi\ , \widehat{S}_\mu\ ] = i\epsilon_{\xi\mu\lambda}\widehat{S}_\lambda$ for angular momentum operators. For bosons the square of the angular momentum operators is given by $\widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2 = (\widehat{N}/2)(\widehat{N}/2 + 1)$, where $\widehat{N} = (\widehat{b}^\dagger\widehat{b} + \widehat{a}^\dagger\widehat{a})$ is the boson total number operator, those for the separate modes being $\widehat{n}_e = \widehat{e}^\dagger\widehat{e}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$). The Schwinger spin operators are the second quantization form of symmetrized one body operators $\widehat{S}_x = \sum_i (|\phi_b(i)\rangle \langle\phi_a(i)| + |\phi_a(i)\rangle \langle\phi_b(i)|)/2$; $\widehat{S}_y = \sum_i (|\phi_b(i)\rangle \langle\phi_a(i)| - |\phi_a(i)\rangle \langle\phi_b(i)|)/2i$; $\widehat{S}_z = \sum_i (|\phi_b(i)\rangle \langle\phi_b(i)| - |\phi_a(i)\rangle \langle\phi_a(i)|)/2$, where the sum i is over the identical bosonic particles. In the case of the two mode EM field the spin angular momentum operators are related to the Stokes parameters.

2.1.2 Bloch Vector and Covariance Matrix

If the density operator for the overall system is $\widehat{\rho}$ then expectation values of the three spin operators $\left\langle \widehat{S}_\xi \right\rangle = Tr(\widehat{\rho}\widehat{S}_\xi)$ ($\xi = x, y, z$) define the *Bloch vector*. Spin squeezing is related to the fluctuation operators $\Delta\widehat{S}_\xi = \widehat{S}_\xi - \left\langle \widehat{S}_\xi \right\rangle$, in terms of which a real, symmetric *covariance matrix* $C(\widehat{S}_\xi, \widehat{S}_\mu)$ ($\xi, \mu = x, y, z$) is defined [14], [6] via

$$\begin{aligned} C(\widehat{S}_\xi, \widehat{S}_\mu) &= (\left\langle \Delta\widehat{S}_\xi\, \Delta\widehat{S}_\mu \right\rangle + \left\langle \Delta\widehat{S}_\mu\, \Delta\widehat{S}_\xi \right\rangle)/2 \\ &= \left\langle \widehat{S}_\xi\, \widehat{S}_\mu + \widehat{S}_\mu\, \widehat{S}_\xi \right\rangle/2 - \left\langle \widehat{S}_\xi \right\rangle \left\langle \widehat{S}_\mu \right\rangle \end{aligned} \tag{2}$$

and whose diagonal elements $C(\widehat{S}_\xi, \widehat{S}_\xi) = \left\langle \Delta\widehat{S}_\xi{}^2 \right\rangle$ gives the variance for the fluctuation operators. The covariance matrix is also *positive definite*. The

variances for the spin operators satisfy the three Heisenberg uncertainty principle reations $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle \left\langle \Delta\widehat{S}_y{}^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{S}_z \right\rangle|^2$; $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle \left\langle \Delta\widehat{S}_z{}^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{S}_x \right\rangle|^2$; $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle \left\langle \Delta\widehat{S}_x{}^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{S}_y \right\rangle|^2$, and spin squeezing is defined via conditions such as $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ with $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$, for $\widehat{S}_x$ being squeezed compared to $\widehat{S}_y$ and so on. Spin squeezing in these components is relevant to tests for entanglement of the modes $\widehat{a}$ and $\widehat{b}$, as will be shown later. Spin squeezing in rotated components is also important, in particular in the so-called *principal* components for which the covariance matrix is diagonal.

2.2 Spin Operators - Multi-Mode Case

As well as spin operators for the simple case of two modes we can also define spin operators in multimode cases involving two sub-systems A and B. For example, there may be two types of bosonic particle involved, each *component* distinguished from the other by having different hyperfine internal states $|A\rangle, |B\rangle$. Each component may be associated with a complete orthonormal set of *spatial mode functions* $\phi_{ai}(\mathbf{r})$ and $\phi_{bi}(\mathbf{r})$, so there will be two sets of modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$, where in the $|\mathbf{r}\rangle$ representation we have $\langle \mathbf{r}\,|\phi_{ai}\rangle = \phi_{ai}(\mathbf{r})\,|A\rangle$ and $\langle \mathbf{r}\,|\phi_{bi}\rangle = \phi_{bi}(\mathbf{r})\,|B\rangle$. Mode orthogonality between A and B modes arises from $\langle A|B\rangle = 0$ rather from the spatial mode functions being orthogonal.

We can define *spin operators* for the combined *multimode* A and B subsystems [15] via

$$
\begin{aligned}
\widehat{S}_x &= \frac{1}{2}\int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) + \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right) \\
\widehat{S}_y &= \frac{1}{2i}\int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) - \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right) \\
\widehat{S}_z &= \frac{1}{2}\int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) - \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right) \qquad (3)
\end{aligned}
$$

where the field operators satisfy the non-zero commutation rules

$$
[\widehat{\Psi}_c(\mathbf{r}), \widehat{\Psi}_d^\dagger(\mathbf{r}')] = \delta_{cd}\,\delta(\mathbf{r}-\mathbf{r}') \qquad c, d = a, b \qquad (4)
$$

It is then easy to show that the standard spin angular momentum conmmutation rules are satisfied. $[\widehat{S}_\xi\ , \widehat{S}_\mu\ ] = i\epsilon_{\xi\mu\lambda}\widehat{S}_\lambda$.

For convenience we can expand the field operators in terms of an orthonormal set of spatial mode functions $\phi_i(\mathbf{r})$. We can choose the spatial mode functions to be the same $\phi_{ai}(\mathbf{r}) = \phi_{bi}(\mathbf{r}) = \phi_i(\mathbf{r})$ (these might be momentum eigenfunctions) and then the field annihilation operators for each component are

$$
\widehat{\Psi}_a(\mathbf{r}) = \sum_i \widehat{a}_i\ \phi_i(\mathbf{r}) \qquad \widehat{\Psi}_b(\mathbf{r}) = \sum_i \widehat{b}_i\ \phi_i(\mathbf{r}) \qquad (5)
$$

These expansions are consistent with the field operator commutation rules (4) based on the usual non-zero mode operator commutation rules $[\widehat{a}_i, \widehat{a}_j^\dagger] = [\widehat{b}_i, \widehat{b}_j^\dagger] = \delta_{ij}$.

By substituting for the field operators we can then express the spin operators in terms of mode operators as

$$\widehat{S}_x = \frac{1}{2}\sum_i \left(\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i\right) \quad \widehat{S}_y = \frac{1}{2i}\sum_i \left(\widehat{b}_i^\dagger \widehat{a}_i - \widehat{a}_i^\dagger \widehat{b}_i\right) \quad \widehat{S}_z = \frac{1}{2}\sum_i \left(\widehat{b}_i^\dagger \widehat{b}_i - \widehat{a}_i^\dagger \widehat{a}_i\right) \tag{6}$$

and it is then easy to confirm that the standard spin angular momentum conmmutation rules are satisfied. $[\widehat{S}_\xi\ ,\widehat{S}_\mu\ ] = i\epsilon_{\xi\mu\lambda}\widehat{S}_\lambda$. We now have both field and mode expressions for spin operators in multimode cases involving two subsystems A and B.

Finally, the *total number of particles* is given by

$$\begin{aligned}
\widehat{N} &= \int d\mathbf{r}\ \left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) + \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right) \\
&= \sum_i \left(\widehat{b}_i^\dagger \widehat{b}_i + \widehat{a}_i^\dagger \widehat{a}_i\right) \\
&= \widehat{N}_b + \widehat{N}_a
\end{aligned} \tag{7}$$

in an obvious notation.

2.3 New Spin Operators and Principal Spin Fluctuations

The covariance matrix has real, non-negative eigenvalues and can be diagonalised via an orthogonal *rotation matrix* $M(-\alpha,-\beta,-\gamma)$ that defines *new spin angular momentum operators* $\widehat{J}_\xi$ $(\xi = x,y,z)$ via

$$\widehat{J}_\xi = \sum_\mu M_{\xi\mu}(-\alpha,-\beta,-\gamma)\widehat{S}_\mu \tag{8}$$

and where

$$\begin{aligned}
C(\widehat{J}_\xi,\widehat{J}_\mu) &= \sum_{\lambda\theta} M_{\xi\lambda}(-\alpha,-\beta,-\gamma)C(\widehat{S}_\lambda,\widehat{S}_\theta)M_{\mu\theta}(-\alpha,-\beta,-\gamma) \\
&= \delta_{\xi\mu}\left\langle \Delta\widehat{J}_\xi^{\,2}\right\rangle
\end{aligned} \tag{9}$$

is the covariance matrix for the new spin angular momentum operators $\widehat{J}_\xi$ $(\xi = x,y,z)$, and which is *diagonal* with the diagonal elements $\left\langle \Delta\widehat{J}_x^{\,2}\right\rangle, \left\langle \Delta\widehat{J}_y^{\,2}\right\rangle$ and $\left\langle \Delta\widehat{J}_z^{\,2}\right\rangle$ giving the so-called *principal spin fluctuations.* The matrix $M(\alpha,\beta,\gamma)$ is parameterised in terms of three Euler angles α,β,γ and is given in [16] (see Eq. (4.43)).

The Bloch vector and spin fluctuations are illustrated in Figure 1. In Fig 1 the Bloch vector and spin fluctuation ellipsoid is shown in terms of the original spin operators $\widehat{S}_\xi$ $(\xi = x,y,z)$

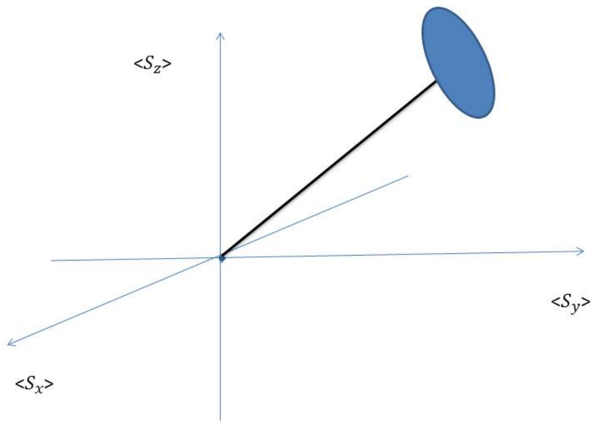

Figure 1. Bloch vector and spin fluctuations shown for original spin operators.

These rules also apply to multimode spin operators as defined in SubSection 2.2.

2.4 Spin Squeezing Definitions

We will begin by considering the case of the spin operators in the most general case. We will also specifically consider spin squeezing for the two new modes and for the multi-mode situation.

2.4.1 Heisenberg Uncertainty Principle and Spin Squeezing

Since the spin operators also satisfy *Heisenberg uncertainty principle* relationships

$$\begin{aligned}\left\langle \Delta\widehat{S}_x{}^2 \right\rangle \left\langle \Delta\widehat{S}_y{}^2 \right\rangle &\geq \frac{1}{4}|\left\langle \widehat{S}_z \right\rangle|^2 \\ \left\langle \Delta\widehat{S}_y{}^2 \right\rangle \left\langle \Delta\widehat{S}_z{}^2 \right\rangle &\geq \frac{1}{4}|\left\langle \widehat{S}_x \right\rangle|^2 \\ \left\langle \Delta\widehat{S}_z{}^2 \right\rangle \left\langle \Delta\widehat{S}_x{}^2 \right\rangle &\geq \frac{1}{4}|\left\langle \widehat{S}_y \right\rangle|^2 \end{aligned} \tag{10}$$

spin squeezing will now be defined for the *spin operators* via condtions such as

$$\begin{aligned}\left\langle \Delta\widehat{S}_x{}^2 \right\rangle &< \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \; and \; \left\langle \Delta\widehat{S}_y{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\ \left\langle \Delta\widehat{S}_y{}^2 \right\rangle &< \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \; and \; \left\langle \Delta\widehat{S}_z{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \\ \left\langle \Delta\widehat{S}_z{}^2 \right\rangle &< \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \; and \; \left\langle \Delta\widehat{S}_x{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \end{aligned} \tag{11}$$

for $\widehat{S}_x$being squeezed compared to $\widehat{S}_y$, and so on.

Note also that the Heisenberg uncertainty principle proof (based on $\left\langle \left(\Delta\widehat{S}_\alpha + i\lambda\Delta\widehat{S}_\beta\right)\left(\Delta\widehat{S}_\alpha - i\lambda\Delta\widehat{S}_\beta\right)\right\rangle \geq 0$ for all real λ) also establishes the general result for all quantum states

$$\left\langle \Delta\widehat{S}_\alpha{}^2 \right\rangle + \left\langle \Delta\widehat{S}_\beta{}^2 \right\rangle \geq |\left\langle \widehat{S}_\gamma \right\rangle| \tag{12}$$

where α, β and γ are x, y and z in cyclic order.

Since the two new mode spin operators defined in Eq. (8) satisy the standard angular momentum operator commutation rules, the usual Heisenberg Uncertainty rules analogous to (10) apply, so that spin squeezing can also exist in the two mode cases involving the *new spin operators* $\widehat{J}_x, \widehat{J}_y$ and $\widehat{J}_z$ as well. These uncertainty principle features also apply to multimode spin operators as defined in SubSection 2.2.

It should be noted that finding spin squeezing for one principal spin operator $\widehat{J}_y$ with respect to another $\widehat{J}_x$ does *not* mean that there is spin squeezing for *any* of the old spin operators $\widehat{S}_x, \widehat{S}_y$ and $\widehat{S}_z$ In the case of the relative phase eigenstate (see SubSection 3.8) $\widehat{J}_y$ is squeezed with respect to $\widehat{J}_x$ - however none of the old spin components are spin squeezed.

2.4.2 Alternative Spin Squeezing Criteria

Other criteria for spin squeezing are also used, for example in the article by Wineland et al [17] . To focus on spin squeezing for $\widehat{S}_z$compared to *any* orthogonal spin operators we can combine the second and third Heisenberg uncertainty principle relationships to give

$$\left\langle \Delta\widehat{S}_z{}^2 \right\rangle \left(\left\langle \Delta\widehat{S}_x{}^2 \right\rangle + \left\langle \Delta\widehat{S}_y{}^2 \right\rangle \right) \geq \frac{1}{4} \left(| \left\langle \widehat{S}_x \right\rangle |^2 + | \left\langle \widehat{S}_y \right\rangle |^2 \right) \tag{13}$$

Then we may define two new spin operators via

$$\widehat{S}_{\perp\,1} = \cos\theta\, \widehat{S}_x + \sin\theta\, \widehat{S}_y \qquad \widehat{S}_{\perp\,2} = -\sin\theta\, \widehat{S}_x + \cos\theta\, \widehat{S}_y \tag{14}$$

where θ corresponds to a rotation angle in the xy plane, and which satisfy the standard angular momentun commutation rules $[\widehat{S}_{\perp\,1}\,, \widehat{S}_{\perp\,2}] = i\widehat{S}_z$, $[\widehat{S}_{\perp\,2}\,, \widehat{S}_z\,] = i\widehat{S}_{\perp\,1}$, $[\widehat{S}_z\,, \widehat{S}_{\perp\,1}] = i\widehat{S}_{\perp\,2}$. It is straightforward to show that $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle + \left\langle \Delta\widehat{S}_y{}^2 \right\rangle = \left\langle \Delta\widehat{S}_{\perp\,1}{}^2 \right\rangle + \left\langle \Delta\widehat{S}_{\perp\,2}{}^2 \right\rangle$ and $| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 = | \left\langle \widehat{S}_x \right\rangle |^2 + | \left\langle \widehat{S}_y \right\rangle |^2$ so that

$$\left\langle \Delta\widehat{S}_z{}^2 \right\rangle \left(\left\langle \Delta\widehat{S}_{\perp\,1}{}^2 \right\rangle + \left\langle \Delta\widehat{S}_{\perp\,2}{}^2 \right\rangle \right) \geq \frac{1}{4} \left(| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 \right) \tag{15}$$

so that *spin squeezing* for $\widehat{S}_z$compared to *any two* orthogonal spin operators such as $\widehat{S}_{\perp\,1}$ or $\widehat{S}_{\perp\,2}$ would be defined as

$$\begin{aligned} \left\langle \Delta\widehat{S}_z{}^2 \right\rangle &< \frac{1}{2}\sqrt{\left(| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 \right)} \\ &\quad \textit{and} \\ \left\langle \Delta\widehat{S}_{\perp\,1}{}^2 \right\rangle + \left\langle \Delta\widehat{S}_{\perp\,2}{}^2 \right\rangle &> \frac{1}{2}\sqrt{\left(| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 \right)} \end{aligned} \tag{16}$$

For spin squeezing in $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle$ we require the *spin squeezing parameter* ξ to satisfy an inequality

$$\xi^2 = \frac{\left\langle \Delta\widehat{S}_z{}^2 \right\rangle}{\left(| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 \right)} < \frac{1}{2\sqrt{\left(| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 \right)}} \sim \frac{1}{N} \tag{17}$$

The last step is an approximation for an N particle state based on the assumption that the Bloch vector lies in the xy plane and close to the Bloch sphere, this situation being the most conducive to detecting the fluctuation $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle$. In this situation $\sqrt{\left(| \left\langle \widehat{S}_{\perp\,1} \right\rangle |^2 + | \left\langle \widehat{S}_{\perp\,2} \right\rangle |^2 \right)}$ is approximately $N/2$. The condition $N\xi^2 < 1$ is sometimes taken as the condition for spin squeezing [18], but it should be noted that this is approximate and Eq. (16) gives the correct expression.

2.4.3 Planar Spin Squeezing

A special case of recent interest is that referred to as *planar squeezing* [19] in which the Bloch vector for a suitable choice of spin operators lies in a *plane* and along one of the *axes*. If this plane is chosen to be the xy plane and the x axis is chosen then $\left\langle \widehat{S}_z \right\rangle = 0$ and $\left\langle \widehat{S}_y \right\rangle = 0$, resulting in only one Heisenberg uncertainty principle relationship where the right side is non-zero, namely $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle \left\langle \Delta\widehat{S}_z{}^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{S}_x \right\rangle|^2$. Combining this with $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle \left\langle \Delta\widehat{S}_y{}^2 \right\rangle \geq 0$ gives $\left(\left\langle \Delta\widehat{S}_y{}^2 \right\rangle + \left\langle \Delta\widehat{S}_x{}^2 \right\rangle\right) \left\langle \Delta\widehat{S}_z{}^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{S}_x \right\rangle|^2$. So the total spin fluctuation *in* the xy plane defined as $\left\langle \Delta\widehat{S}_{para}{}^2 \right\rangle = \left\langle \Delta\widehat{S}_y{}^2 \right\rangle + \left\langle \Delta\widehat{S}_x{}^2 \right\rangle$ will be squeezed compared to the spin fluctuation perpendicular to the xy plane given by $\left\langle \Delta\widehat{J}_{perp}{}^2 \right\rangle = \left\langle \Delta\widehat{S}_z{}^2 \right\rangle$ if

$$\left\langle \Delta\widehat{S}_{para}{}^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \; and \; \left\langle \Delta\widehat{S}_{perp}{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \tag{18}$$

By minimising $\left\langle \Delta\widehat{S}_{para}{}^2 \right\rangle$ whilst satisfying the constraints $\left\langle \widehat{S}_z \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0$ a spin squeezed state is found that satisfies (18) with $\left\langle \Delta\widehat{S}_{para}{}^2 \right\rangle \,\tilde{}\, J^{2/3}$, $\left\langle \Delta\widehat{S}_{perp}{}^2 \right\rangle \sim J^{4/3}$, $|\left\langle \widehat{S}_x \right\rangle| \sim J$ for large $J = N/2$ [19]. The Bloch vector is on the Bloch sphere.

2.4.4 Spin Squeezing in Multi-Mode Cases

Since the multi-mode spin operators defined in Eq.(3) satisy the standard angular momentum operator commutation rules, the usual Heisenberg Uncertainty rules analogous to (10) apply, so that spin squeezing can also exist in the multi-mode case as well. Thus

$$\begin{aligned}
\left\langle \Delta\widehat{S}_x{}^2 \right\rangle &< \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \; and \; \left\langle \Delta\widehat{S}_y{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\
\left\langle \Delta\widehat{S}_y{}^2 \right\rangle &< \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \; and \; \left\langle \Delta\widehat{S}_z{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \\
\left\langle \Delta\widehat{S}_z{}^2 \right\rangle &< \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \; and \; \left\langle \Delta\widehat{S}_x{}^2 \right\rangle > \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \qquad (19)
\end{aligned}$$

for $\widehat{S}_x$being squeezed compared to $\widehat{S}_y$, and so on.

Sinmilar alternative criteria to (16) can also be obtained, for example for $\widehat{S}_z$ being squeezed compared to *any two* orthogonal spin operators such as $\widehat{S}_{\perp\,1}$ or $\widehat{S}_{\perp\,2}$ defined similarly to (14).

2.5 Rotation Operators and New Modes

2.5.1 Rotation Operators

The new spin operators are also related to the original spin operators via a *unitary rotation operator* $\widehat{R}(\alpha, \beta, \gamma)$ parameterised in terms of Euler angles so that

$$\widehat{J}_\xi = \widehat{R}(\alpha, \beta, \gamma)\, \widehat{S}_\xi\, \widehat{R}(\alpha, \beta, \gamma)^{-1} \tag{20}$$

where

$$\widehat{R}(\alpha, \beta, \gamma) = \widehat{R}_z(\alpha)\widehat{R}_y(\beta)\widehat{R}_z(\gamma) \tag{21}$$

with $\widehat{R}_\xi(\phi) = \exp(i\phi\widehat{S}_\xi)$ describing a rotation about the ξ axis anticlockwise through an angle ϕ. Details for the rotation operators and matrices are set out in [6]. Note that Eq. (20) specifies a rotation of the vector spin operator rather than a rotation of the axes, so $\widehat{J}_\xi$ ($\xi = x, y, z$) are the components of the rotated vector spin operator with respect to the original axes.

2.5.2 New Mode Operators

We can also see that the new spin operators are related to *new mode operators* $\widehat{c}$ and $\widehat{d}$ via

$$\widehat{J}_x = (\widehat{d}^\dagger\widehat{c} + \widehat{c}^\dagger\widehat{d})/2 \qquad \widehat{J}_y = (\widehat{d}^\dagger\widehat{c} - \widehat{c}^\dagger\widehat{d})/2i \qquad \widehat{J}_z = (\widehat{d}^\dagger\widehat{d} - \widehat{c}^\dagger\widehat{c})/2 \tag{22}$$

where

$$\widehat{c} = \widehat{R}(\alpha, \beta, \gamma)\, \widehat{a}\, \widehat{R}(\alpha, \beta, \gamma)^{-1} \qquad \widehat{d} = \widehat{R}(\alpha, \beta, \gamma)\, \widehat{b}\, \widehat{R}(\alpha, \beta, \gamma)^{-1} \tag{23}$$

For the bosonic case a straight-forward calculation gives the new mode operators as

$$\begin{aligned}
\widehat{c} &= \exp(\frac{1}{2}i\gamma)\left(\cos(\frac{\beta}{2})\,\exp(\frac{1}{2}i\alpha)\,\widehat{a} + \sin(\frac{\beta}{2})\,\exp(-\frac{1}{2}i\alpha)\,\widehat{b}\right) \\
\widehat{d} &= \exp(-\frac{1}{2}i\gamma)\left(-\sin(\frac{\beta}{2})\,\exp(\frac{1}{2}i\alpha)\,\widehat{a} + \cos(\frac{\beta}{2})\,\exp(-\frac{1}{2}i\alpha)\,\widehat{b}\right)
\end{aligned} \tag{24}$$

and it is easy to then check that $\widehat{c}$ and $\widehat{d}$ satisfy the expected non-zero bosonic commutation rules are $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{c}$ or $\widehat{d}$) and that the *total boson number operator* is $\widehat{N} = (\widehat{d}^\dagger\widehat{d} + \widehat{c}^\dagger\widehat{c})$. As $\widehat{N}$ is invariant under unitary rotation operators it follows that $\widehat{J}_x^2 + \widehat{J}_y^2 + \widehat{J}_z^2 = (\widehat{N}/2)(\widehat{N}/2 + 1)$.

2.5.3 New Modes

The new mode operators correspond to *new single particle states* $|\phi_c\rangle$, $|\phi_d\rangle$ where

$$\begin{aligned}|\phi_c\rangle &= \exp(-\frac{1}{2}i\gamma)\left(\cos(\frac{\beta}{2})\exp(-\frac{1}{2}i\alpha)\,|\phi_a\rangle + \sin(\frac{\beta}{2})\exp(\frac{1}{2}i\alpha)\,|\phi_b\rangle\right)\\ |\phi_d\rangle &= \exp(\frac{1}{2}i\gamma)\left(-\sin(\frac{\beta}{2})\exp(-\frac{1}{2}i\alpha)\,|\phi_a\rangle + \cos(\frac{\beta}{2})\exp(\frac{1}{2}i\alpha)\,|\phi_b\rangle\right)\end{aligned} \tag{25}$$

These are two orthonormal quantum superpositions of the original single particle states $|\phi_a\rangle$, $|\phi_b\rangle$, and as such represent an *alternative choice* of modes that could be realised experimentally.

Eqs. (24) can be inverted to give the old mode operators via

$$\begin{aligned}\widehat{a} &= \exp(-\frac{1}{2}i\alpha)\left(\cos(\frac{\beta}{2})\exp(-\frac{1}{2}i\gamma)\,\widehat{c} - \sin(\frac{\beta}{2})\exp(+\frac{1}{2}i\gamma)\,\widehat{d}\right)\\ \widehat{b} &= \exp(+\frac{1}{2}i\alpha)\left(\sin(\frac{\beta}{2})\exp(\frac{1}{2}i\gamma)\,\widehat{c} + \cos(\frac{\beta}{2})\exp(-\frac{1}{2}i\gamma)\,\widehat{d}\right)\end{aligned} \tag{26}$$

2.6 Old and New Modes - Coherence Terms

The general non-entangled state for modes $\widehat{a}$ and $\widehat{b}$ is given by

$$\widehat{\rho} = \sum_R P_R\,\widehat{\rho}_R^A \otimes \widehat{\rho}_R^B \tag{27}$$

and as a consequence of the requirement that $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are physical states for modes $\widehat{a}$ and $\widehat{b}$ satisying the super-selection rule, it follows that

$$\begin{aligned}\langle(\widehat{a})^n\rangle_c &= Tr(\widehat{\rho}_R^A(\widehat{a})^n) = 0 \qquad \left\langle(\widehat{a}^\dagger)^n\right\rangle_c = Tr(\widehat{\rho}_R^A(\widehat{a}^\dagger)^n) = 0\\ \left\langle(\widehat{b})^m\right\rangle_d &= Tr(\widehat{\rho}_R^B(\widehat{b})^m) = 0 \qquad \left\langle(\widehat{b}^\dagger)^m\right\rangle_d = Tr(\widehat{\rho}_R^B(\widehat{b}^\dagger)^m) = 0\end{aligned} \tag{28}$$

Thus *coherence* terms are zero.

For our two-mode case we have also seen that the original choice of modes with annihilation operators $\widehat{a}$ and $\widehat{b}$ may be replaced by new modes with annihilation operators $\widehat{c}$ and $\widehat{d}$. Since the new modes are associated with new spin operators $\widehat{J}_\xi$ ($\xi = x, y, z$) for which the covariance matrix is diagonal and where the diagonal elements give the variances, it is therefore also relevant to consider entanglement for the case where the sub-systems are modes $\widehat{c}$ and $\widehat{d}$, rather than $\widehat{a}$ and $\widehat{b}$. Consequently we also consider general non-entangled states for modes $\widehat{c}$ and $\widehat{d}$ in which the density operator is of the same form as (27), but with $\widehat{\rho}_R^A \rightarrow \widehat{\rho}_R^C$ and $\widehat{\rho}_R^B \rightarrow \widehat{\rho}_R^D$. Results analogous to (28) apply in this case, but with $\widehat{a} \rightarrow \widehat{c}$ and $\widehat{b} \rightarrow \widehat{d}$.

2.7 Quantum Correlation Functions and Spin Measurements

Finally, we note that the principal spin fluctuations can be related to *quantum correlation functions*. For example, it is easy to show that

$$\begin{aligned}\left\langle \Delta\widehat{S}_x{}^2 \right\rangle &= \frac{1}{4}\left(\left\langle (\widehat{b}^\dagger)^2(\widehat{a})^2 \right\rangle + \left\langle (\widehat{a}^\dagger)^2(\widehat{b})^2 \right\rangle + 2\left\langle \widehat{b}^\dagger\widehat{a}^\dagger\widehat{a}\widehat{b} \right\rangle + \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle + \left\langle \widehat{a}^\dagger\widehat{a} \right\rangle\right) \\ &\quad -\frac{1}{4}\left(\left\langle (\widehat{b}^\dagger\widehat{a} \right\rangle^2 + \left\langle (\widehat{a}^\dagger\widehat{b} \right\rangle^2 + 2\left\langle (\widehat{b}^\dagger\widehat{a} \right\rangle \left\langle (\widehat{a}^\dagger\widehat{b} \right\rangle\right) \end{aligned} \tag{29}$$

showing that $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle$ is related to various first and second order quantum correlation functions. These can be measured experimentally and are given theoretically in terms of phase space integrals involving distribution functions to represent the density operator and phase space variables to represent the mode annihilation, creation operators.

3 Spin Squeezing Test for Entanglement

With the general non-entangled state now required to be such that the density operators for the individual sub-systems must represent quantum states that conform to the super-selection rule, the consequential link between entanglement in two mode bosonic systems and spin squeezing can now be established. We first consider spin squeezing for the original spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ and entangled states of the original modes $\widehat{a}$, $\widehat{b}$. and then for the principal spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$ and entangled states of the related new modes $\widehat{c}$, $\widehat{d}$. We show [1] that spin squeezing in *any* spin component is a *sufficiency test* for entanglement of the two modes involved. Examples of entangled states that are not spin squeezed and states that are not entangled nor spin squeezed for one choice of mode sub-systems, but are entangled and spin squeezed for another choice are then presented.

3.1 Spin Squeezing and Entanglement - Old Modes

Firstly, the *mean* for a Hermitian operator $\widehat{\Omega}$ in a mixed state

$$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R \tag{30}$$

is the *average* of means for separate components

$$\left\langle \widehat{\Omega} \right\rangle = \sum_R P_R \left\langle \widehat{\Omega} \right\rangle_R \tag{31}$$

where $\left\langle \widehat{\Omega} \right\rangle_R = Tr(\widehat{\rho}_R \, \widehat{\Omega})$.

Secondly, the *variance* for a Hermitian operator $\widehat{\Omega}$ in a mixed state is always *never less* than the the *average* of the variances for the separate components (see [20])

$$\left\langle \Delta\widehat{\Omega}^2 \right\rangle \geq \sum_R P_R \left\langle \Delta\widehat{\Omega}_R{}^2 \right\rangle_R \tag{32}$$

where $\left\langle \Delta\widehat{\Omega}^2 \right\rangle = Tr(\widehat{\rho} \, \Delta\widehat{\Omega}^2)$ with $\Delta\widehat{\Omega} = \widehat{\Omega} - \left\langle \widehat{\Omega} \right\rangle$ and $\left\langle \Delta\widehat{\Omega}^2 \right\rangle_R = Tr(\widehat{\rho}_R \, \Delta\widehat{\Omega}_R{}^2)$ with $\Delta\widehat{\Omega}_R = \widehat{\Omega} - \left\langle \widehat{\Omega} \right\rangle_R$. To prove this result we have using (31) both for $\widehat{\Omega}$ and $\widehat{\Omega}^2$

$$\begin{aligned}
\left\langle \Delta\widehat{\Omega}^2 \right\rangle &= \left\langle \widehat{\Omega}^2 \right\rangle - \left\langle \widehat{\Omega} \right\rangle^2 \\
&= \sum_R P_R \left(\left\langle \widehat{\Omega}^2 \right\rangle_R - \left\langle \widehat{\Omega} \right\rangle_R^2 \right) + \sum_R P_R \left\langle \widehat{\Omega} \right\rangle_R^2 - \left(\sum_R P_R \left\langle \widehat{\Omega} \right\rangle_R \right)^2 \\
&= \sum_R P_R \left\langle \Delta\widehat{\Omega}_R^2 \right\rangle_R + \sum_R P_R \left\langle \widehat{\Omega} \right\rangle_R^2 - \left(\sum_R P_R \, | \left\langle \widehat{\Omega} \right\rangle_R | \right)^2
\end{aligned} \tag{33}$$

The variance result (32) follows because the sum of the last two terms is always ≥ 0 using the result (**175**) **in Appendix 2 of paper 1**, with $C_R = \left\langle \widehat{\Omega} \right\rangle_R^2$, $\sqrt{C_R} = |\left\langle \widehat{\Omega} \right\rangle_R|$- which are real and positive.

3.1.1 Mean Values for $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$

Next, we find the *mean values* of the spin operators for the product state $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$

$$\begin{aligned} \left\langle \widehat{S}_x \right\rangle_R &= \frac{1}{2}(\left\langle \widehat{b}^\dagger \right\rangle_R \langle \widehat{a} \rangle_R + \langle \widehat{a}^\dagger \rangle_R \left\langle \widehat{b} \right\rangle_R) = 0 \\ \left\langle \widehat{S}_y \right\rangle_R &= \frac{1}{2i}(\left\langle \widehat{b}^\dagger \right\rangle_R \langle \widehat{a} \rangle_R - \langle \widehat{a}^\dagger \rangle_R \left\langle \widehat{b} \right\rangle_R) = 0 \end{aligned} \quad (34)$$

and

$$\left\langle \widehat{S}_z \right\rangle_R = \frac{1}{2}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R - \langle \widehat{a}^\dagger \widehat{a} \rangle_R) \quad (35)$$

for SSR compliant sub-system states using (28), and thus using (31) the *overall mean* values for the *separable* state is

$$\left\langle \widehat{S}_x \right\rangle = 0 \qquad \left\langle \widehat{S}_y \right\rangle = 0 \quad (36)$$

and

$$\left\langle \widehat{S}_z \right\rangle = \sum_R P_R \frac{1}{2}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R - \langle \widehat{a}^\dagger \widehat{a} \rangle_R)) \quad (37)$$

Hence if either $\left\langle \widehat{S}_x \right\rangle \neq 0$ or $\left\langle \widehat{S}_y \right\rangle \neq 0$ the state must be entangled. This may be called the *Bloch vector* test. This result will also have later significance.

3.1.2 Variances for $\widehat{S}_x$ and $\widehat{S}_y$

Next we calculate $\left\langle \Delta \widehat{S}_x^2 \right\rangle_R$, $\left\langle \Delta \widehat{S}_y^2 \right\rangle_R$ and $\left\langle \widehat{S}_x \right\rangle_R$, $\left\langle \widehat{S}_y \right\rangle_R$, $\left\langle \widehat{S}_z \right\rangle_R$ for the case of the *separable state* (27) where $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$. From Eqs. (1) we find that

$$\begin{aligned} \widehat{S}_x^2 &= \frac{1}{4}((\widehat{b}^\dagger)^2(\widehat{a})^2 + \widehat{b}^\dagger \widehat{b} \widehat{a} \widehat{a}^\dagger + \widehat{a}^\dagger \widehat{a} \widehat{b} \widehat{b}^\dagger + (\widehat{b})^2(\widehat{a}^\dagger)^2) \\ \widehat{S}_y^2 &= -\frac{1}{4}((\widehat{b}^\dagger)^2(\widehat{a})^2 - \widehat{b}^\dagger \widehat{b} \widehat{a} \widehat{a}^\dagger - \widehat{a}^\dagger \widehat{a} \widehat{b} \widehat{b}^\dagger + (\widehat{b})^2(\widehat{a}^\dagger)^2) \end{aligned} \quad (38)$$

so that on taking the trace with $\widehat{\rho}_R$ and using Eqs. (28) we get after applying the commutation rules $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$)

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle_R &= \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \\ \left\langle \widehat{S}_y^2 \right\rangle_R &= \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \end{aligned} \quad (39)$$

Hence using (34) for $\left\langle \widehat{S}_x \right\rangle_R$ and $\left\langle \widehat{S}_y \right\rangle_R$ we see finally that the variances are

$$\begin{aligned}
\left\langle \Delta \widehat{S}_x^2 \right\rangle_R &= \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \\
\left\langle \Delta \widehat{S}_y^2 \right\rangle_R &= \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \qquad (40)
\end{aligned}$$

and therefore from Eq. (32)

$$\begin{aligned}
\left\langle \Delta \widehat{S}_x^2 \right\rangle &\geq \sum_R P_R \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \\
\left\langle \Delta \widehat{S}_y^2 \right\rangle &\geq \sum_R P_R \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \qquad (41)
\end{aligned}$$

Now using (37) for $\left\langle \widehat{S}_z \right\rangle$ we see that

$$\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \leq \sum_R P_R \frac{1}{4}|(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R - \langle \widehat{a}^\dagger \widehat{a} \rangle_R))| \leq \sum_R P_R \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R)) \quad (42)$$

and thus for any non-entangled state of modes $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned}
& \left\langle \Delta \widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\
\geq\ & \sum_R P_R \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) - \sum_R P_R \frac{1}{4}(\left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger \widehat{a} \rangle_R)) \\
\geq\ & \sum_R P_R \frac{1}{2}(\langle \widehat{a}^\dagger \widehat{a} \rangle_R \left\langle \widehat{b}^\dagger \widehat{b} \right\rangle_R) \\
\geq\ & 0 \qquad (43)
\end{aligned}$$

Similar final steps show that $\left\langle \Delta \widehat{S}_y^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \geq 0$ for all non-entangled state of modes $\widehat{a}$ and $\widehat{b}$.

This shows that for the general non-entangled state with modes $\widehat{a}$ and $\widehat{b}$ as the sub-systems, the variances for two of the principal spin fluctuations $\left\langle \Delta \widehat{S}_x^2 \right\rangle$ and $\left\langle \Delta \widehat{S}_y^2 \right\rangle$ are *both* greater than $\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$, and hence there is no spin squeezing for $\widehat{S}_x$ compared to $\widehat{S}_y$ (or vice versa). Note that as $|\left\langle \widehat{S}_y \right\rangle| = 0$, the quantity $\sqrt{\left(|\left\langle \widehat{S}_{\perp 1} \right\rangle|^2 + |\left\langle \widehat{S}_{\perp 2} \right\rangle|^2\right)}$ is the same as $|\left\langle \widehat{S}_z \right\rangle|$, so the alternative criterion in Eq. (16) is the same as that in Eq. (11) which is used here.

It is easy to see from (36) that

$$\left\langle \Delta \widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \geq 0 \qquad \left\langle \Delta \widehat{S}_y^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \geq 0 \qquad (44)$$

for any non-entangled state of modes $\widehat{a}$ and $\widehat{b}$. This completes the set of inequalities for the variances of $\widehat{S}_x$ and $\widehat{S}_y$. These last inequalities are of course trivially true and amount to no more than showing that the variances $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ and $\left\langle \Delta\widehat{S}_y^2 \right\rangle$ are not negative.

3.1.3 Variance for $\widehat{S}_z$

For the other principal spin fluctuation we find that for separable states

$$\left\langle \Delta\widehat{S}_z^2 \right\rangle_R = \frac{1}{4}(\left\langle \left(\widehat{b}^\dagger\widehat{b} - \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right) \left(\widehat{b}^\dagger\widehat{b} - \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right) \right\rangle_R + \left\langle \left(\widehat{a}^\dagger\widehat{a} - \langle \widehat{a}^\dagger\widehat{a} \rangle_R\right) \left(\widehat{a}^\dagger\widehat{a} - \langle \widehat{a}^\dagger\widehat{a} \rangle_R\right) \right\rangle_R \tag{45}$$

so that using (32)

$$\left\langle \Delta\widehat{S}_z^2 \right\rangle \geq \sum_R P_R \frac{1}{4}(\left\langle \left(\widehat{b}^\dagger\widehat{b} - \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right)^2 \right\rangle_R + \left\langle \left(\widehat{a}^\dagger\widehat{a} - \langle \widehat{a}^\dagger\widehat{a} \rangle_R\right)^2 \right\rangle_R \tag{46}$$

From Eq. (36) it follows that

$$\begin{aligned} &\left\langle \Delta\widehat{S}_z^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \\ \geq\ & \sum_R P_R \frac{1}{4}(\left\langle \left(\widehat{b}^\dagger\widehat{b} - \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right)^2 \right\rangle_R + \left\langle \left(\widehat{a}^\dagger\widehat{a} - \langle \widehat{a}^\dagger\widehat{a} \rangle_R\right)^2 \right\rangle_R \\ \geq\ & 0 \end{aligned} \tag{47}$$

Similarly $\left\langle \Delta\widehat{S}_z^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \geq 0$. Again, these results are trivial and just show that the variances are non-negative.

3.1.4 No Spin Squeezing for Separable States

So overall, we have for the general non-entangled state of modes $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned} \left\langle \Delta\widehat{S}_x{}^2 \right\rangle &\geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \; and \; \left\langle \Delta\widehat{S}_y{}^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\ \left\langle \Delta\widehat{S}_y{}^2 \right\rangle &\geq \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \; and \; \left\langle \Delta\widehat{S}_z{}^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_x \right\rangle| \\ \left\langle \Delta\widehat{S}_z{}^2 \right\rangle &\geq \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \; and \; \left\langle \Delta\widehat{S}_x{}^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_y \right\rangle| \end{aligned} \tag{48}$$

The first result tells us that for *any* non-entangled state of modes $\widehat{a}$ and $\widehat{b}$ the spin operator $\widehat{S}_x$ is *not* squeezed compared to $\widehat{S}_y$ (or vice-versa). The same is also true for the other pairs of spin operators, as we will now see.

3.1.5 Spin Squeezing Tests for Entanglement

The key value of these results is the *spin squeezing test* for *entanglement.* We see that from the first inequality in (48) for separable states, that *if* for a quantum

state we find that

$$\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \qquad or \quad \left\langle \Delta\widehat{S}_{y}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \tag{49}$$

then the state *must* be entangled for modes $\widehat{a}$ and $\widehat{b}$. Thus we only need to have spin squeezing in *either* of $\widehat{S}_x$ with respect to $\widehat{S}_y$ *or* vice-versa to demonstrate entanglement. Note that one cannot have *both* $\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle|$ and$\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle|$ etc. due to the Heisenberg uncertainty principle.

Because $\left\langle \widehat{S}_x\right\rangle_\rho = \left\langle \widehat{S}_y\right\rangle_\rho = 0$ the second and third results in (48) merely show that $\left\langle \Delta\widehat{S}_x{}^2\right\rangle \geq 0$, $\left\langle \Delta\widehat{S}_y{}^2\right\rangle \geq 0$ and $\left\langle \Delta\widehat{S}_z{}^2\right\rangle \geq 0$ for SSR compliant non-entangled states, it may be thought that no conclusion follows regarding the spin squeezing involving $\widehat{S}_z$ for entangled states. This is not the case. *If* for a given state we find that

$$\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{x}\right\rangle| \qquad or \quad \left\langle \Delta\widehat{S}_{z}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{x}\right\rangle| \tag{50}$$

or

$$\left\langle \Delta\widehat{S}_{z}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{y}\right\rangle| \qquad or \quad \left\langle \Delta\widehat{S}_{x}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{y}\right\rangle| \tag{51}$$

then the state *must* be entangled. This is because if any of these situations apply then *at least one* of $\left\langle \widehat{S}_x\right\rangle_\rho$ or $\left\langle \widehat{S}_y\right\rangle_\rho$ must be non-zero. But as we have seen from (36) both of the quantities are zero in non-entangled states. Thus we only need to have spin squeezing in *either* of $\widehat{S}_z$ with respect to $\widehat{S}_y$ *or* vice-versa or spin squeezing in *either* of $\widehat{S}_z$ with respect to $\widehat{S}_x$ *or* vice-versa to demonstrate entanglement.

Hence the general conclusion stated in [1], that spin squeezing in *any* spin operator $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ shows that the state must be entangled for modes $\widehat{a}$ and $\widehat{b}$.. The presence of spin squeezing is a conclusive test for entanglement. Note that the reverse is not true - there are many entangled states that are *not* spin squeezed. A notable example is the particular *binomial* state $|\Phi\rangle = ((\widehat{a}+\widehat{b})^\dagger/\sqrt{2})^N/\sqrt{N!}\ |0\rangle$ for which $\left\langle \widehat{S}_x\right\rangle_\rho = N/2$, $\left\langle \widehat{S}_y\right\rangle_\rho = \left\langle \widehat{S}_z\right\rangle_\rho = 0$ and $\left\langle \Delta\widehat{S}_y{}^2\right\rangle_\rho = \left\langle \Delta\widehat{S}_z{}^2\right\rangle_\rho = N/4$, $\left\langle \Delta\widehat{S}_x{}^2\right\rangle_\rho = 0$ (see [6]) The spin fluctuations in $\widehat{S}_y$ and $\widehat{S}_z$ correspond to the *standard quantum limit*.

This is a *key result* for two mode entanglement. *All* spin squeezed states are *entangled*. We emphasise again that the converse is not true. *Not all* entangled two mode states are *spin squeezed*. This important distinction is not always recognised - entanglement and spin squeezing are two *distinct* features of a two mode quantum state that do not always occur together.

For the two orthogonal spin operator components (14) in the xy plane $\widehat{S}_{\perp 1}$ and $\widehat{S}_{\perp 1}$ it is then straightforward to show that

$$If \quad \left\langle \Delta\widehat{S}_{\perp 1}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \tag{52}$$

or

$$If \quad \left\langle \Delta\widehat{S}_{\perp 2}^{2}\right\rangle < \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \tag{53}$$

that is, if $\widehat{S}_{\perp 1}$ is squeezed compared to $\widehat{S}_{\perp 2}$ or vice versa - then the state must be entangled. Spin squeezing in *any* of the spin operator component in the xy plane will demonstrate entanglement.

3.1.6 Significance of Spin Squeezing Test

The spin squeezing test for two mode systems was based on the general form (27) for all *separable* states together with the requirement that the sub-system density operators $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ were compliant with the *local* particle number *SSR*. From the point of view of a *supporter* for applying the local particle number SSR if the result of an experiment is that spin squeezing has occurred, the immediate conclusion is that the state is entangled. On the other hand from the point of view of a *sceptic* about being required to apply the local particle number SSR for the sub-system states, such a sceptic would draw different conclusions from an experiment that demonstrated spin squeezing. They would immediately point out that in this case spin squeezing is *not* a test for entanglement. However, as we will now see these conclusions are still of some interest.

To discuss this it is convenient to divide possible *mathematical* forms for the density operator into categories. Considering *all* general two mode quantum states that are compliant with the *global* particle number SSR, we may first divide such quantum states into three categories, as set out in Table 1.

REGION	**OVERALL**	**SUB-SYSTEM**	**CATEGORY**
	QUANTUM STATE	**QUANTUM STATE**	
A	$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$	Both $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are local	* **Separable**
		particle number SSR compliant	
B	$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$	Neither $\widehat{\rho}_R^A$ nor $\widehat{\rho}_R^B$ is local	* **Separable but**
		particle number SSR compliant	**non-local [3];**
			* **Entangled [1]**
C	$\widehat{\rho} \neq \sum_R P_R\, \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$	N/A	* **Entangled**

Table I. Categories of two mode quantum states.

The regions referred to are shown in Figure 2. All authors would regard the quantum states in Region A as being separable and those in Region C as being entangled - it is only those in Region B where the category is disputed.

Those such as [1] (local SSR supporters) who require local particle number SSR compliance for each sub-system state would classify the overall state as entangled, those who do not require this (local SSR sceptics) such as [3] would classify the overall state as separable but non-local. Note that no further sub-classification is needed.

In **SubSection 3.2∂ of paper I** we show that if states of the form (56) are globally SSR compliant, then *both* the sub-system states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are local particle number SSR compliant *in general*. However, we point out that there are special matched choices for both $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ along with the P_R, where neither $\widehat{\rho}_R^A$ nor $\widehat{\rho}_R^B$ is local particle number SSR compliant even though $\widehat{\rho}$ is global particle number SSR compliant. But the case where just *one* of $\widehat{\rho}_R^A$ or $\widehat{\rho}_R^B$ is non SSR compliant does not occur, so Region B does not need to be sub-divided along these lines.

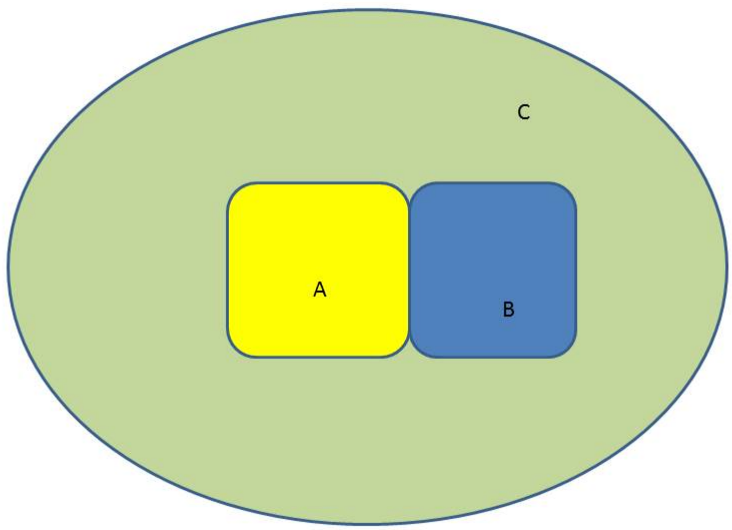

Figure 2. Categories of two mode quantum states that are global particle number SSR compliant. The regions A, B and C are described in Table I.

Now let us consider quantum states for which $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$ and $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$. Such states are clearly not spin squeezed. Firstly, we

know that *all* states in Region A satisfy these inequalities. However, *some* states in Region B and *some* states in Region C may also satisfy these inequalities. In Figure 3 the quantum states in Region B that satisfy these inequalities are depicted as lying in Region D, those in Region C that do so are depicted as lying in Region F. Hence, if we find that the quantum state is such that spin squeezing *does* occur (as in the test of (59)) we can definitely say that it does *not* lie in Regions A, D or F. It must therefore be located in Regions E or L. The question is - Does this determine whether the state is entangled or not according to the *supporters* of applying the local SSR as in the definition of entanglement used in the present paper? The answer is that it does. This is because the quantum state must be located within either of Regions B or C, since these regions include E and L respectively. In both cases it would be *entangled* according to the definition used here [1] (see Table 1).

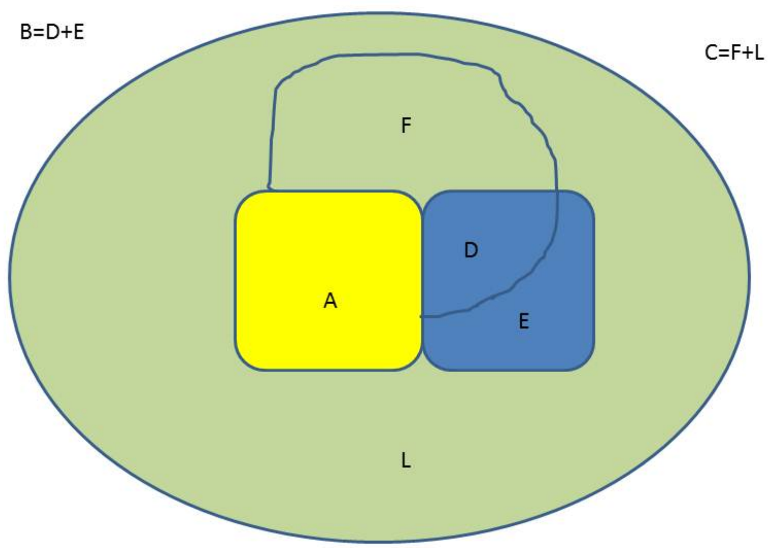

Figure 3. Categories of two mode quantum states satisfying inequalities $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$ and $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$ These regions are A, D, F. Set unions are denoted $+$. Referring to Figure 2, $B = D \oplus E$ and $C = F \oplus L$.

However, the *sceptics* of applying the local SSR would draw a different conclusion from the experiment that demonstrated spin squeezing (as in the test of (59)). They would agree that the mathematics shows that a state in Region

A could not demonstrate spin squeezing. Nor by assumption could states in Regions D or F. This means that the state must lie in either Region L or Region E. So from the point of view of the sceptic, *either* the state is *entangled* (if it lies in Region L) *or* the sub-system states in *all* separable states (Region E) do not comply with the local particle number *SSR*. The sceptic's conclusion is clearly interesting - in the first case the quantum state is entangled, and hence may demonstrate other non-classical features, and in the second case the possibility exists of finding sub-systems in states that have the unexpected feature in non-relativistic many body physics of having coherences between states with differing particle number. If there was a *second* experimental test that could show that the state was not entangled, then this would demonstrate the existence of quantum states (sub-systems are themselves possible quantum systems) in which the particle number SSR breaks down.

The second experiment would seem to require a test for entanglement which is *necessary* as well as being sufficient - the latter alone being usually the case for entanglement tests. Such criteria and measurements are a challenge, but not impossible even though we have not met this challenge in these two papers. Thus, *in principle* there could be a *pair* of experiments that give evidence of entanglement, *or* failure of the Super Selection Rule. For such investigations to be possible, the use of entanglement criteria that *do* invoke the local super-selection rules is *also* required. Such tests are the focus of these two papers, though here our primary reason is because we consider applying the local particle number SSR is required by the physics of non-relativistic quantum many body systems involving massive particles.

3.1.7 Inequality for $|\left\langle \widehat{S}_z \right\rangle|$

Of the results for a *non-entangled* physical state for modes $\widehat{a}$ and $\widehat{b}$ we will later find it particularly important to consider the first of (48)

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \quad and \quad \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \tag{54}$$

This is because we can show that for any quantum state

$$|\left\langle \widehat{S}_z \right\rangle| = |\left\langle \frac{1}{2}(\widehat{n}_b - \widehat{n}_a) \right\rangle| \leq \frac{1}{2}(|\left\langle \widehat{n}_b \right\rangle| + |\left\langle \widehat{n}_a \right\rangle|) = \frac{1}{2}\left\langle \widehat{N} \right\rangle \tag{55}$$

there is an inequality involving $|\left\langle \widehat{S}_z \right\rangle|$ and the mean number of bosons $\left\langle \widehat{N} \right\rangle$ in the two mode system. Note that there *are* some entangled states (see SubSection 3.6) for which $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ and $\left\langle \Delta\widehat{S}_y^2 \right\rangle$ are both greater than $\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$, since all that has been proven is that for *all* non-entangled states we must have *both* $\left\langle \Delta\widehat{S}_x^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ *and* $\left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$.

Hence we may conclude that spin squeezing in any of the original spin fluctuations $\widehat{S}_x$, $\widehat{S}_y$ or $\widehat{S}_z$ requires the quantum state to be entangled for the modes

$\widehat{a}$ and $\widehat{b}$ as the sub-systems. Similarly, we may conclude that spin squeezing in any of the principal spin fluctuations $\widehat{J}_x$, $\widehat{J}_y$ or $\widehat{J}_z$ requires the quantum state to be entangled for the modes $\widehat{c}$ and $\widehat{d}$ as the sub-systems, these modes being associated with the principal spin fluctuations via Eq. (22). Although finding spin squeezing tells us that the state is entangled, there are however no simple relationships between the measures of entanglement and those of spin squeezing, so the linkage is essentially a qualitative one. For general quantum states, measures of entanglement for the specific situation of two sub-systems (bi-partite entanglement) are reviewed in [21].

3.2 Spin Squeezing and Entanglement - New Modes

It is also of some interest to consider spin squeezing for the new spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$ with the new modes $\widehat{c}$ and $\widehat{d}$ as the sub-systems, where these spin operators are associated with a diagonal covariance matrix. The definition of spin squeezing in this case is set out analogous to that in Eq.(19). In this case the general non-entangled state for the *new* modes is given by

$$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R^C \otimes \widehat{\rho}_R^D \tag{56}$$

with the $\widehat{\rho}_R^C$ and $\widehat{\rho}_R^D$ representing physical states for modes $\widehat{c}$ and $\widehat{d}$, and where results analogous to Eqs. (28) apply. The same treatment applies as for spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ with the modes $\widehat{a}$ and $\widehat{b}$ as the sub-systems and leads to the result for a *non-entangled* state of modes $\widehat{c}$ and $\widehat{d}$

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle \geq \frac{1}{2} |\left\langle \widehat{J}_z \right\rangle| \quad and \quad \left\langle \Delta\widehat{J}_y^2 \right\rangle \geq \frac{1}{2} |\left\langle \widehat{J}_z \right\rangle \tag{57}$$

showing that neither $\widehat{J}_x$ or $\widehat{J}_y$ is spin squeezed for the general non-entangled state for modes $\widehat{c}$ and $\widehat{d}$ given in Eq. (24). We also have

$$\left\langle \widehat{J}_x \right\rangle = \sum_R P_R \left\langle \widehat{J}_x \right\rangle_R = 0 \qquad \left\langle \widehat{J}_y \right\rangle = \sum_R P_R \left\langle \widehat{J}_y \right\rangle_R = 0 \tag{58}$$

so all the results analogous to Eqs. (48) also follow. Following similar arguements as in SubSection 3.1 we may also conclude that spin squeezing in *any* of the original spin fluctuations requires the quantum state to be entangled when the original modes $\widehat{c}$ and $\widehat{d}$ are the sub-systems. Thus the *entanglement test* is

$$If \quad \left\langle \Delta\widehat{J}_x^2 \right\rangle < \frac{1}{2} |\left\langle \widehat{J}_z \right\rangle| \qquad or \quad \left\langle \Delta\widehat{J}_y^2 \right\rangle < \frac{1}{2} |\left\langle \widehat{J}_z \right\rangle| \tag{59}$$

or

$$If \quad \left\langle \Delta\widehat{J}_y^2 \right\rangle < \frac{1}{2} |\left\langle \widehat{J}_x \right\rangle| \qquad or \quad \left\langle \Delta\widehat{J}_z^2 \right\rangle < \frac{1}{2} |\left\langle \widehat{J}_x \right\rangle| \tag{60}$$

or

$$If \quad \left\langle \Delta\widehat{J}_z^2 \right\rangle < \frac{1}{2} |\left\langle \widehat{J}_y \right\rangle| \qquad or \quad \left\langle \Delta\widehat{J}_x^2 \right\rangle < \frac{1}{2} |\left\langle \widehat{J}_y \right\rangle| \tag{61}$$

then we have an entangled state for the original modes $\widehat{c}$ and $\widehat{d}$.

The result (58) also provides a *Bloch vector* entanglement test - if either $\left\langle \widehat{S}_x \right\rangle \neq 0$ or $\left\langle \widehat{S}_y \right\rangle \neq 0$ the state must be entangled.

Hence we have seen that spin squeezing - either of the new or original spin operators requires entanglement of the new or original modes. Which spin operators to consider depends on which pairs of modes are being tested for entanglement.

3.3 Spin Squeezing and Entanglement - Multi-Mode Case

As we have seen the multi-mode case involves a set of n modes with annihilation operators $\widehat{a}_i$ for bosons with hyperfine component A, and another set of n modes with annihilation operators $\widehat{b}_i$ for bosons with hyperfine component B. Since entanglement implies a clear choice of what sub-systems are to be entangled, there are numerous choices possible here for the present multi-mode case. *Case 1* involves two sub-systems, one consisting of all the $\widehat{a}_i$ modes as sub-system A and the other consisting of all the $\widehat{b}_i$ modes as sub-system B. *Case 2* involves $2n$ sub-systems, the Ai th containing the mode $\widehat{a}_i$ and the Bi th containing the mode $\widehat{b}_i$. *Case 3* involves n sub-systems, the ith containiing the two modes $\widehat{a}_i$ and $\widehat{b}_i$. These three cases relate to entanglement causing interactions in differing circumstances. Case 1 might apply to cases where separable states can be created with all the $\widehat{a}_i$ modes coupled together to produce states $\widehat{\rho}_R^A$ and the $\widehat{b}_i$ modes coupled together to produce states $\widehat{\rho}_R^B$. Case 2 might apply cases where separable states can be created with the $\widehat{a}_i$ and all the $\widehat{b}_i$ modes independent of each together to produce states $\widehat{\rho}_R^{a(i)} \otimes \widehat{\rho}_R^{b(i)}$.Case 3 might apply cases where separable states can be created with the $\widehat{a}_i$ and the matching $\widehat{b}_i$ modes coupled together to produce states $\widehat{\rho}_R^{ab(i)}$. Cases 2 and 3 will be discussed further in SubSection 4.4 dealing with the entanglement test introduced by Sorensen et al [13].

The density operators for *separable* states in the three cases will be of the form

$$
\begin{aligned}
\widehat{\rho}_{sep} &= \sum_R P_R \widehat{\rho}_R \\
\widehat{\rho}_R &= \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B \qquad Case \quad 1 && (62) \\
\widehat{\rho}_R &= \widehat{\rho}_R^{a(1)} \otimes .. \otimes \widehat{\rho}_R^{a(i)} .. \otimes \widehat{\rho}_R^{a(n)} \otimes \widehat{\rho}_R^{b(1)} \otimes .. \otimes \widehat{\rho}_R^{b(n)} \qquad Case \quad 2 && (63) \\
\widehat{\rho}_R &= \widehat{\rho}_R^{ab(1)} \otimes \widehat{\rho}_R^{ab(2)} \otimes .. \otimes \widehat{\rho}_R^{ab(i)} .. \otimes \widehat{\rho}_R^{ab(n)} \qquad Case \quad 3 && (64) \\
& && (65)
\end{aligned}
$$

Discussion of whether there is a spin squeezing test for Case 1 in the multi-mode case involves a generalisation of the theory set out in SubSection 3.1. There is a *Bloch vector* entanglement test, in that if either of $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero, then the state is entangled. We also find that spin squeezing in *any* spin component requires the state to be entangled, thus generalising the spin

squeezing test to the *multi-mode* case, for *two* sub-systems consisting of all the modes $\widehat{a}_i$ and al the modes $\widehat{b}_i$ The details are covered in Appendix 11.

For Case 2 a spin squeezing test for entanglement also be obtained. The test is again that *spin squeezing* in *any* spin component $\widehat{S}_x, \widehat{S}_y$ or $\widehat{S}_z$ confirms entanglement of the $2n$ sub-systems consisting of *single* modes $\widehat{a}_i$ and $\widehat{b}_i$. Furthermore, there is also a *Bloch vector* entanglement test, in that if either of $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero, then the state is entangled. As these systems can have quantum *states* with large numbers N of bosonic particles, it can be said that entanglement in an N particle *system* has occurred if spin squeezing is found. The proof of these tests is set out in SubSection 12.1 of Appendix 12.

For Case 3 there is also a spin squeezing test for entanglement, but it is restricted. Here the test is that *spin squeezing* in $\widehat{S}_z$ confirms entanglement of the n sub-systems consisting of *pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$, but the test is *restricted* to the situation where exactly *one boson* occupies each mode pair. No spin squeezing test was found for the other spin operators, nor was a Bloch vector entanglement test obtained. The proof of this result is set out in SubSection 12.2 of Appendix 12. That no general spin squeezing test for entanglement exists can be shown by a counter-example. If all the N bosons occupied one mode pair $\widehat{a}_i$ and $\widehat{b}_i$, and the quantum state $\widehat{\rho}_R^{ab(i)}$ for this pair corresponded to the *relative phase eigenstate* with phase $\theta_p = 0$ (see SubSection 3.8) then although the overall state is separable, spin squeezing in $\widehat{S}_y$ compared to $\widehat{S}_z$ occurs (with $\left\langle \Delta \widehat{S}_y^2 \right\rangle = \frac{1}{4} + \frac{1}{8} \ln N$, $\left\langle \Delta \widehat{S}_z^2 \right\rangle = \left(\frac{1}{6} - \frac{\pi^2}{64} \right) N^2$ and $\left\langle \widehat{S}_x \right\rangle = N \frac{\pi}{8}$. Thus there is a situation where a *non-entangled* state for sub-systems consisting of *mode pairs* is spin squeezed, so spin squeezing does *not* always confirm entanglement.

As in the previous two mode cases, having established in multi-mode cases that spin squeezing requires entanglement a further question then is: Does entanglement automatically lead to spin squeezing? The answer is no, since cases where the quantum state is entangled but not spin squeezed can be found - an example is given in the previous paragraph . Thus in general, spin squeezing and entanglement are *not equivalent*.- they do not occur *together* for all states. Spin squeezing is a *sufficient* condition for entanglement, it is not a *necessary* condition.

3.4 Bloch Vector Entanglement Test

We have seen for the general non-entangled states of modes $\widehat{c}$ and $\widehat{d}$ or of modes $\widehat{a}$ and $\widehat{b}$ that

$$\left\langle \widehat{J}_x \right\rangle = 0 \qquad \left\langle \widehat{J}_y \right\rangle = 0 \tag{66}$$

$$\left\langle \widehat{S}_x \right\rangle = 0 \qquad \left\langle \widehat{S}_y \right\rangle = 0 \tag{67}$$

Hence the two mode *Bloch vector entanglement* tests

$$\begin{aligned}\left\langle \widehat{J}_x \right\rangle &\neq 0 \qquad or \qquad \left\langle \widehat{J}_y \right\rangle \neq 0 \\ \left\langle \widehat{S}_x \right\rangle &\neq 0 \qquad or \qquad \left\langle \widehat{S}_y \right\rangle \neq 0 \end{aligned} \tag{68}$$

for modes $\widehat{c}$ and $\widehat{d}$ or of modes $\widehat{a}$ and $\widehat{b}$. The same Bloch vector test also applies in the *multi-mode case* for Case 1, where there are just two sub-systems each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$ and in Case 2, where there are $2n$ sub-systems consisting of all the modes $\widehat{a}_i$ and all the modes $\widehat{b}_i$.

From Eqs. (22) and (1) these results are equivalent to

$$\left\langle \widehat{d}\widehat{c}^{\dagger} \right\rangle = 0 \qquad \left\langle \widehat{c}\widehat{d}^{\dagger} \right\rangle = 0 \tag{69}$$

$$\left\langle \widehat{b}\widehat{a}^{\dagger} \right\rangle = 0 \qquad \left\langle \widehat{a}\widehat{b}^{\dagger} \right\rangle = 0 \tag{70}$$

Hence we find further *tests* for *entangled states* of modes $\widehat{c}$ and $\widehat{d}$ or of modes $\widehat{a}$ and $\widehat{b}$

$$|\left\langle \widehat{d}\widehat{c}^{\dagger} \right\rangle|^2 > 0 \qquad |\left\langle \widehat{c}\widehat{d}^{\dagger} \right\rangle|^2 > 0 \tag{71}$$

$$|\left\langle \widehat{b}\widehat{a}^{\dagger} \right\rangle|^2 > 0 \qquad |\left\langle \widehat{a}\widehat{b}^{\dagger} \right\rangle|^2 > 0 \tag{72}$$

As we will see in Section 4, these tests are particular cases with $m = n = 1$ of the simpler entanglement test in Eq. (165) that applies for the situation in the present paper where non-entangled states are required to satisfy the super-selection rule.

3.5 Entanglement Test for Number Difference and Sum

There is also a further spin squeezing test involving the operator $\widehat{S}_z$, which is equal to half the *number difference* $\frac{1}{2}(\widehat{n}_b - \widehat{n}_a)$. We note that simultaneous eigenststes of $\widehat{n}_a$ and $\widehat{n}_b$ exist, which are also eigenstates of the total number operator. For such states the variances $\langle \Delta\widehat{n}_a^2 \rangle$, $\langle \Delta\widehat{n}_b^2 \rangle$, $\left\langle \Delta\widehat{S}_z^2 \right\rangle$ and $\left\langle \Delta\widehat{N}^2 \right\rangle$ are all zero, which does not suggest that useful general inequalities for these variances would be found. However, a useful entanglement test - which does not require SSR compliance can be found. For the variance of $\widehat{S}_z$ in a separable state we have

$$\begin{aligned}\left\langle \Delta\widehat{S}_z^2 \right\rangle &\geq \sum_R P_R \left\langle \Delta\widehat{S}_z^2 \right\rangle_R = \sum_R P_R(\left\langle \widehat{S}_z^2 \right\rangle_R - \left\langle \widehat{S}_z \right\rangle_R^2) \\ &= \frac{1}{4}\sum_R P_R(\left\langle \widehat{n}_b^2 \right\rangle_R + \left\langle \widehat{n}_a^2 \right\rangle_R - 2\left\langle \widehat{n}_b \right\rangle_R \left\langle \widehat{n}_a \right\rangle_R - \left\langle \widehat{n}_b \right\rangle_R^2 - \left\langle \widehat{n}_a \right\rangle_R^2 + 2\left\langle \widehat{n}_b \right\rangle_R \left\langle \widehat{n}_a \right\rangle_R) \\ &= \frac{1}{4}\sum_R P_R(\left\langle \Delta\widehat{n}_b^2 \right\rangle_R + \left\langle \Delta\widehat{n}_a^2 \right\rangle_R) \end{aligned} \tag{73}$$

For such a separable state we also find

$$\left\langle \Delta\widehat{N}^2 \right\rangle \geq \sum_R P_R(\langle \Delta\widehat{n}_b^2 \rangle_R + \langle \Delta\widehat{n}_a^2 \rangle_R) \tag{74}$$

This leads to the useful if somewhat qualitative test that if we have a state with a *large* fluctuation in the total boson number and a *small* fluctuation in the number difference, then it must be an entangled state. If it was separable and $\left\langle \Delta\widehat{N}^2 \right\rangle$ is large, then $\left\langle \Delta\widehat{S}_z^2 \right\rangle$ must also be large. There is also the converse test - if we have a state with a small fluctuation in the total boson number and a large fluctuation in the number difference, then it must be an entangled state.

3.6 Entangled States that are Non Spin-Squeezed - NOON State

One such example is the generalised N boson *NOON state* defined as

$$\begin{aligned}
\widehat{\rho} &= |\Phi\rangle\langle\Phi| \\
|\Phi\rangle &= \cos\theta\,\frac{(\widehat{a}^\dagger)^N}{\sqrt{N!}}\,|0\rangle + \sin\theta\,\frac{(\widehat{b}^\dagger)^N}{\sqrt{N!}}\,|0\rangle \\
&= \cos\theta\left|\frac{N}{2},-\frac{N}{2}\right\rangle + \sin\theta\left|\frac{N}{2},+\frac{N}{2}\right\rangle
\end{aligned} \tag{75}$$

which is an entangled state for modes $\widehat{a}$ and $\widehat{b}$ in all cases except where $\cos\theta$ or $\sin\theta$ is zero. In the last form the state is expressed in terms of the eigenstates for $(\overrightarrow{\widehat{S}})^2$ and $\widehat{S}_z$, as detailed in [6].

A straight-forward calculation gives

$$\begin{aligned}
\left\langle \widehat{S}_x \right\rangle &= 0 \qquad \left\langle \widehat{S}_y \right\rangle = 0 \qquad \left\langle \widehat{S}_z \right\rangle = -\frac{N}{2}\cos 2\theta \\
\left\langle \Delta\widehat{S}_x^2 \right\rangle &= \frac{N}{4} \qquad \left\langle \Delta\widehat{S}_y^2 \right\rangle = \frac{N}{4} \qquad \left\langle \Delta\widehat{S}_z^2 \right\rangle = \frac{N^2}{4}(1-\cos^2 2\theta)
\end{aligned} \tag{76}$$

for $N > 1$, so that using the criteria for spin squeezing given in Eq. (11) we see that as $\left\langle \Delta\widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \geq 0$, etc, and hence spin squeezing does not occur for this entangled state.

3.7 Non-Entangled States that are Non Spin Squeezed - Binomial State

Of course from the previous section *any* non entangled state is definitely not spin squeezed. A specific example illustrating this is the N boson binomial state given by

$$\begin{aligned}
\widehat{\rho} &= |\Phi\rangle\langle\Phi| \\
|\Phi\rangle &= \frac{(-\widehat{c}^\dagger)^N}{\sqrt{N!}}\,|0\rangle
\end{aligned} \tag{77}$$

where $\widehat{c}$ and $\widehat{d}$ are given by Eqs. (24) with Euler angles $\alpha = -\pi + \chi$, $\beta = -2\theta$ and $\gamma = -\pi$, we find that

$$\begin{aligned}
\widehat{c} &= -\cos\theta \, \exp(\frac{1}{2}i\chi)\,\widehat{a} - \sin\theta \, \exp(-\frac{1}{2}i\chi)\,\widehat{b} = -\widehat{a}_1 \\
\widehat{d} &= \sin\theta \, \exp(\frac{1}{2}i\chi)\,\widehat{a} - \cos\theta \, \exp(-\frac{1}{2}i\chi)\,\widehat{b} = -\widehat{a}_2
\end{aligned} \tag{78}$$

where the mode operators $\widehat{a}_1$ and $\widehat{a}_2$ are as defined in [6] (see Eqs. (53) therein). The new spin angular momentum operators $\widehat{J}_\xi$ $(\xi = x, y, z)$ are the same as those defined in [6] (see Eqs. (64) therein) and in [6] it has been shown (see Eq. (60) therein) for the same binomial state as in (77) that

$$\begin{aligned}
\left\langle \widehat{J}_x \right\rangle &= 0 \qquad \left\langle \widehat{J}_y \right\rangle = 0 \qquad \left\langle \widehat{J}_z \right\rangle = -\frac{N}{2} \\
\left\langle \Delta \widehat{J}_x^2 \right\rangle &= \frac{N}{4} \qquad \left\langle \Delta \widehat{J}_y^2 \right\rangle = \frac{N}{4} \qquad \left\langle \Delta \widehat{J}_z^2 \right\rangle = 0
\end{aligned} \tag{79}$$

(see Eqs. (162) and (176) therein). Hence the binomial state is not spin squeezed since $\left\langle \Delta \widehat{J}_x^2 \right\rangle = \left\langle \Delta \widehat{J}_y^2 \right\rangle = \frac{1}{2}|\left\langle \widehat{J}_z \right\rangle|$. It is of course a *minimum uncertainty state* with spin fluctuations at the *standard quantum limit*. Here $\sqrt{\left\langle \Delta \widehat{J}_{x,y}^2 \right\rangle}/|\left\langle \widehat{J}_z \right\rangle| = 1/\sqrt{N}$. Clearly, it is a non-entangled state for modes $\widehat{c}$ and $\widehat{d}$, being the product of a number state for mode $\widehat{c}$ with the vacuum state for mode $\widehat{d}$.

Note that from the point of view of the original modes $\widehat{a}$ and $\widehat{b}$, this is an entangled state. so the question is: Is it a spin squeezed state with respect to the original spin operators $\widehat{S}_\xi$ $(\xi = x, y, z)$? The Bloch vector and variances for this binomial state are given in [6] (see Eq. (163) in the main paper and Eq. (410) in the Appendix). The results include:

$$\begin{aligned}
\left\langle \widehat{S}_z \right\rangle &= -\frac{N}{2}\cos 2\theta \\
\left\langle \Delta \widehat{S}_x^2 \right\rangle &= \frac{N}{4}(\cos^2 2\theta \, \cos^2\chi + \sin^2\chi) \qquad \left\langle \Delta \widehat{S}_y^2 \right\rangle = \frac{N}{4}(\cos^2 2\theta \, \sin^2\chi + \cos^2\chi)
\end{aligned} \tag{80}$$

This gives $\left\langle \Delta \widehat{S}_x^2 \right\rangle \left\langle \Delta \widehat{S}_y^2 \right\rangle - \frac{1}{4}|\left\langle \widehat{S}_z \right\rangle|^2 = \frac{1}{16}N^2(\cos^2 2\theta - 1)^2 \cos^2\chi \sin^2\chi \geq 0$ as required for the Heisenberg uncertainty principle. With $\chi = 0$ we have $\left\langle \Delta \widehat{S}_x^2 \right\rangle = \frac{N}{4}\cos^2 2\theta$ and $\left\langle \Delta \widehat{S}_y^2 \right\rangle = \frac{N}{4}$, whilst $\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| = \frac{N}{4}|\cos 2\theta|$. As $\left\langle \Delta \widehat{S}_x^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ there is spin squeezing in $\widehat{S}_x$ for this entangled state of modes $\widehat{a}$ and $\widehat{b}$, though not of course for the new spin operator $\widehat{J}_x$ since this state is non-entangled for modes $\widehat{c}$ and $\widehat{d}$. This example illustrates the need to carefully define spin squeezing and entanglement in terms of related sets of spin operators and modes. The same state is entangled with respect to one choice

of modes - and spin squeezing occurs, whilst it is non-entangled with respect to another set of modes - and no spin squeezing occurs.

To summarise - with a physically based definition of non-entangled states for bosonic systems with two modes (related to the principal spin operators that have a diagonal covariance matrix) being the sub-systems and with a criterion for spin squeezing that focuses on these principal spin fluctuations, it seen that whilst non-entangled states are never spin squeezed and therefore although entanglement is a necessary condition for spin squeezing, it is not a sufficient one. There are entangled states that are not spin squeezed. Furthermore, as there is no simple quantitative links between measures of spin squeezing and those for entanglement, the mere presence of spin squeezing only demonstrates the qualitative result that the quantum state is entangled. Nevertheless, for high precision measurements based on spin operators where the primary emphasis is on how much spin squeezing can be achieved, knowing that entangled states are needed provides an impetus for studying such states and how they might be produced.

3.8 Entangled States that are Spin Squeezed - Relative Phase Eigenstate

As an example of an entangled state that is spin squeezed we consider the relative phase eigenstate $\left|\frac{N}{2},\theta_p\right\rangle$ for a two mode system in which there are N bosons. For modes with annihilation operators $\widehat{a}$, $\widehat{b}$ the *relative phase eigenstate* is defined as

$$\left|\frac{N}{2},\theta_p\right\rangle = \frac{1}{\sqrt{N+1}}\sum_{k=-N/2}^{N/2}\exp(ik\theta_p)\,\frac{(\widehat{a}^{\dagger})^{N/2-k}}{\sqrt{(N/2-k)!}}\frac{(\widehat{b}^{\dagger})^{N/2+k}}{\sqrt{(N/2+k)!}}\left|0\right\rangle \tag{81}$$

where the relative phase $\theta_p = p(2\pi/(N{+}1))$ with $p = -N/2, -N/2{+}1, ..., +N/2$, is an eigenvalue of the relative phase Hermitian operator of the type introduced by Barnett and Pegg [22] (see [6] and references therein). Note that the eigenvalues form a quasi-continuum over the range $-\pi$ to $+\pi$, with a small separation between neighboring phases of $O(1/N)$. The relative phase state is consistent with the super-selection rule and is an entangled state for modes $\widehat{a}$, $\widehat{b}$. The Bloch vector for spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ is given by (see [6])

$$\left\langle \widehat{S}_x\right\rangle = N\,\frac{\pi}{8}\cos\theta_p \qquad \left\langle \widehat{S}_y\right\rangle = -N\,\frac{\pi}{8}\sin\theta_p \qquad \left\langle \widehat{S}_z\right\rangle = 0 \tag{82}$$

but the covariance matrix (see Eq. (178) in [6]) is non-diagonal.

3.8.1 New Spin Operators

It is more instructive to consider spin squeezing in terms of new spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$ for which the covariance matrix is diagonal. The new spin operators

are related to the original spin operators via

$$\begin{aligned}\widehat{J}_x &= \widehat{S}_z \\ \widehat{J}_y &= \sin\theta_p\,\widehat{S}_x + \cos\theta_p\,\widehat{S}_y \\ \widehat{J}_z &= -\cos\theta_p\,\widehat{S}_x + \sin\theta_p\,\widehat{S}_y\end{aligned} \tag{83}$$

corresponding to the transformation in Eq. (8) with Euler angles $\alpha = -\pi + \theta_p$, $\beta = -\pi/2$ and $\gamma = -\pi$.

3.8.2 Bloch Vector and Covariance Matrix

The Bloch vector and covariance matrix for spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$ are given by (see Eqs. (180), (181) in [6] - note that the $C(\widehat{J}_y, \widehat{J}_y)$ element is incorrect in Eq. (181))

$$\left\langle \widehat{J}_x \right\rangle = 0 \qquad \left\langle \widehat{J}_y \right\rangle = 0 \qquad \left\langle \widehat{J}_z \right\rangle = -N\,\frac{\pi}{8} \tag{84}$$

and

$$\begin{aligned}&\begin{bmatrix} C(\widehat{J}_x,\widehat{J}_x) & C(\widehat{J}_x,\widehat{J}_y) & C(\widehat{J}_x,\widehat{J}_z) \\ C(\widehat{J}_y,\widehat{J}_x) & C(\widehat{J}_y,\widehat{J}_y) & C(\widehat{J}_y,\widehat{J}_z) \\ C(\widehat{J}_z,\widehat{J}_y) & C(\widehat{J}_z,\widehat{J}_y) & C(\widehat{J}_z,\widehat{J}_z) \end{bmatrix} \\ \doteqdot &\begin{bmatrix} \frac{1}{12}N^2 & 0 & 0 \\ 0 & \frac{1}{4}+\frac{1}{8}\ln N & 0 \\ 0 & 0 & \left(\frac{1}{6}-\frac{\pi^2}{64}\right)N^2 \end{bmatrix} \qquad N \gg 1\end{aligned} \tag{85}$$

With $\left\langle \Delta\widehat{J}_x^2 \right\rangle = \frac{1}{12}N^2$, $\left\langle \Delta\widehat{J}_y^2 \right\rangle = \frac{1}{4} + \frac{1}{8}\ln N$ and $\left\langle \Delta\widehat{J}_z^2 \right\rangle = \left(\frac{1}{6} - \frac{\pi^2}{64}\right)N^2$ and the only non-zero Bloch vector component being $\left\langle \widehat{J}_z \right\rangle = -N\,\frac{\pi}{8}$ it is easy to see that $\left\langle \Delta\widehat{J}_x^2 \right\rangle \left\langle \Delta\widehat{J}_y^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{J}_z \right\rangle|^2$ as required by the Heisenberg Uncertainty Principle. The principal spin fluctuations in both $\widehat{J}_x$ and $\widehat{J}_z$ are comparable to the length of the Bloch vector and no spin squeezing occurs in either of these components. However, spin squeezing occurs in that $\widehat{J}_y$ is squeezed with respect to $\widehat{J}_x$ - $\left\langle \Delta\widehat{J}_y^2 \right\rangle$ only increases as $\frac{1}{8}\ln N$ whilst $\frac{1}{2}|\left\langle \widehat{J}_z \right\rangle|$ increases as $\frac{\pi}{16}N$ for large N. Hence the relative phase state satisfies the test in Eq. (49) to demonstate entanglement for modes $\widehat{c}$, $\widehat{d}$. Here $\sqrt{\left\langle \Delta\widehat{J}_{,y}^2 \right\rangle}/|\left\langle \widehat{J}_z \right\rangle| \sim \sqrt{\ln N}/N$ which indicates that the Heisenberg limit is being reached.

Note that *none* of the old spin components are spin squeezed. As shown in Ref. [6] $\left\langle \Delta\widehat{S}_x^2 \right\rangle = \left(\frac{1}{6} - \frac{\pi^2}{64}\right)\cos^2\theta_p\,N^2$, $\left\langle \Delta\widehat{S}_y^2 \right\rangle = \left(\frac{1}{6} - \frac{\pi^2}{64}\right)\sin^2\theta_p\,N^2$ and $\left\langle \Delta\widehat{S}_z^2 \right\rangle = \frac{1}{12}N^2$, along with $\left\langle \widehat{S}_x \right\rangle = N\,\frac{\pi}{8}\cos\theta_p$ $\left\langle \widehat{S}_y \right\rangle = -N\,\frac{\pi}{8}\sin\theta_p$ $\left\langle \widehat{S}_z \right\rangle = 0$. All variances are of order N^2 whilst the non-zero means are only of order N. Hence spin squeezing in one of the principal spin operators does *not* imply spin squeezing in any of the original spin operators. This is relevant to spin squeezing tests for entanglement of the *original* modes.

3.8.3 New Modes Operators

To confirm that the relative phase state is in fact an entangled state for modes $\widehat{c}$, $\widehat{d}$ as well as for the original modes $\widehat{a}$, $\widehat{b}$, we note that the new mode operators $\widehat{c}$, $\widehat{d}$ are given in in Eq. (24) with Euler angles $\alpha = -\pi + \theta_p$, $\beta = -\pi/2$ and $\gamma = -\pi$. The old mode operators are given in Eq. (26) and with these Euler angles we have

$$\begin{aligned}
\widehat{a} &= -\exp(\frac{1}{2}i\theta_p)\frac{1}{\sqrt{2}}\left(\widehat{c} - \widehat{d}\right) \\
\widehat{b} &= -\exp(-\frac{1}{2}i\theta_p)\frac{1}{\sqrt{2}}\left(\widehat{c} + \widehat{d}\right)
\end{aligned} \tag{86}$$

This enables us to write the phase state in terms of new mode operators $\widehat{c}$, $\widehat{d}$ as

$$\begin{aligned}
\left|\frac{N}{2},\theta_p\right\rangle &= \frac{1}{\sqrt{N+1}}\left(\frac{-1}{\sqrt{2}}\right)^N \sum_{k=-N/2}^{N/2}\sum_{r=-N/4+k/2}^{N/4-k/2}\sum_{s=-N/4-k/2}^{N/4+k/2} \\
&\times \frac{1}{\sqrt{(N/2-k)!}}\frac{1}{\sqrt{(N/2+k)!}}(-1)^{N/4-k/2+r} \\
&\times \frac{(N/2-k)!}{(N/4-k/2-r)!(N/4-k/2+r)!}\frac{(N/2+k)!}{(N/4+k/2-s)!(N/4+k/2+s)!} \\
&\times (\widehat{c}^{\dagger})^{N/2-(r+s)}\,(\widehat{d}^{\dagger})^{N/2+(r+s)}\,|0\rangle
\end{aligned} \tag{87}$$

We see that the expression does not depend explicitly on the relative phase θ_p when written in terms of the new mode unnormalised Fock states $(\widehat{c}^{\dagger})^{N/2-(r+s)}\,(\widehat{d}^{\dagger})^{N/2+(r+s)}\,|0\rangle$. This pure state is a linear combination of product states of the form $|N/2-m\rangle_c \otimes |N/2+m\rangle_d$ for various m - each of which is an N boson state and an eigenstate for $\widehat{J}_z$ with eigenvalue m, and therefore is an entangled state for modes $\widehat{c}$, $\widehat{d}$ which is compatible with the global super-selection rule. Note that there cannot just be a single term m involved, otherwise the variance for $\widehat{J}_z$ would be zero instead of $\left(\frac{1}{6} - \frac{\pi^2}{64}\right)N^2$. We will return to the relative phase state again in SubSection 4.1.

4 Other Spin Operator Tests for Entanglement

In this Section we examine a number of previously stated entanglement tests involving spin operators. It turns out that *many* of the tests do confirm entanglement for massive bosons according to the SSR and symmetrisation principle compliant definition as it is defined here, though not always for the reasons given in their original proofs. Importantly, in some cases for massive bosons the tests can be made more general.

There are a number of inequalities involving the *spin operators* that have previously been derived for testing whether a state for a system of identical bosons is entangled. These are *not* always associated with criteria for spin squeezing - which involve the variances and mean values of the spin operators. Also, some of these inequalities are *not* based on the requirement that the density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ in the expression for a non-entangled state conform to the *super-selection rule* that prohibits quantum superpositions of single mode states with differing numbers of bosons (which was invoked because they represent possible quantum states for the separate modes - *local particle number SSR compliance*). Only generic quantum properties of the sub-system density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ were used in the derivations. In contrast, our results are based in effect on a *stricter criterion* as to what constitutes a *separable state*, so of course we obtain *new* entanglement tests. However, entanglement tests which were based on *not requiring* SSR compliance for $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ will *also* confirm entanglement when SSR compliance is *required*. This outcome occurs in the SubSection 4.1 in the case of the Hillery spin variance entanglement test. It also occurs in SubSection 4.2 for the entanglement test in (117) involving spin operators for two mode sub-systems, in SubSection 4.3 for the entanglement test in (129) involving mean values of powers of local spin operators, and in two entanglement tests (152), (154) in SubSection 4.5 that involve variances of two mode spin operators.

Other entanglement tests have been proposed whose proofs were based on forms of the density operator for non-entangled states that are *not* consistent with the *symmetrisation principle*. The sub-systems were regarded as labelled individual particles, and strictly speaking, this should only apply to systems of *distinguishable* particles. These include the spin squeezing in the total spin operator $\widehat{S}_z$ test (133) in SubSection 4.4. In that SubSection we show that the original proof in [13] can be modified to treat *identical* particles but now with distinguishable pairs of modes as the sub-systems, but the proof requires that the separable states are *restricted* to one boson per mode pair. However, in SubSection 3.1 we have already shown that for two mode systems in which SSR compliance applies spin squeezing in $\widehat{S}_z$ demonstrates two mode entanglement. Also, in SubSection 3.3 we showed that in multi-mode cases modes associated with two different internal (hyperfine) components, that spin squeezing in $\widehat{S}_z$ also shows entanglement occurs in two situations - one where there are *two* sub-systems each just consisting of modes associated with the same internal component (Case 1), the second where *each* mode counts as a separate sub-system (Case 2). Thus the spin squeezing in $\widehat{S}_z$ test does demonstrate en-

tanglement for identical massive bosons, though not for the reasons given in the original proof. These new SSR compliant proofs now confirm the spin squeezing in $\widehat{S}_z$ as a valid test for entanglement in two component or two mode BECs.

4.1 Hillery et al 2006

4.1.1 Hillery Spin Variance Entanglement Test

An entanglement test in which local particle number SSR compliance is ignored is presented in the paper by Hillery and Zubairy [23] entitled "Entanglement conditions for two-mode states". The paper actually dealt with EM field modes, and the density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ for photon modes allowed for coherences between states with differing photon numbers. A discussion of SSR for the case of photons is presented in Paper I, **in SubSection 3.2**. Hence the conditions in Eq. (28) were not applied.

We will now derive the Hillery spin variance inequalities involving $\left\langle \Delta\widehat{S}_x \right\rangle^2$, $\left\langle \Delta\widehat{S}_y \right\rangle^2$ by applying a similar treatment to that in SubSection 3.1, but now ignoring local particle number SSR compliance. It is found that for the original spin operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ and modes $\widehat{a}$ and $\widehat{b}$

$$\begin{aligned}
\left\langle \widehat{S}_x^2 \right\rangle_R &= \frac{1}{4}(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger\widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger\widehat{a} \rangle_R \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R) \\
&\quad + \frac{1}{4}(\left\langle (\widehat{b}^\dagger)^2 \right\rangle_R \langle (\widehat{a})^2 \rangle_R + \left\langle (\widehat{b})^2 \right\rangle_R \langle (\widehat{a}^\dagger)^2 \rangle) \\
\left\langle \widehat{S}_y^2 \right\rangle_R &= \frac{1}{4}(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger\widehat{a} \rangle_R) + \frac{1}{2}(\langle \widehat{a}^\dagger\widehat{a} \rangle_R \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R) \\
&\quad - \frac{1}{4}(\left\langle (\widehat{b}^\dagger)^2 \right\rangle_R \langle (\widehat{a})^2 \rangle_R + \left\langle (\widehat{b})^2 \right\rangle_R \langle (\widehat{a}^\dagger)^2 \rangle) \qquad (88)
\end{aligned}$$

where terms such as $\left\langle (\widehat{b}^\dagger)^2 \right\rangle_R$ and $\langle (\widehat{a})^2 \rangle_R$ previously shown to be zero have been retained. Note that in [23] the operators $\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ constructed from the EM field mode operators as in Eq. (1) would be related to Stokes parameters Hence

$$\begin{aligned}
&\left\langle \widehat{S}_x^2 \right\rangle_R + \left\langle \widehat{S}_y^2 \right\rangle_R \\
&= \frac{1}{2}(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger\widehat{a} \rangle_R) + (\langle \widehat{a}^\dagger\widehat{a} \rangle_R \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R) \\
&= \frac{1}{2}(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R (\langle \widehat{a}^\dagger\widehat{a} \rangle_R + 1) + \frac{1}{2}(\langle \widehat{a}^\dagger\widehat{a} \rangle_R (\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + 1)) \qquad (89)
\end{aligned}$$

where the terms $\left\langle (\widehat{b}^\dagger)^2 \right\rangle_R$, ..., $\langle (\widehat{a}^\dagger)^2 \rangle$ cancel out. This is the same as before.

However,

$$\begin{aligned}
\left\langle \widehat{S}_x \right\rangle_R &= \frac{1}{2}(\left\langle \widehat{b}^\dagger \right\rangle_R \langle \widehat{a} \rangle_R + \langle \widehat{a}^\dagger \rangle_R \left\langle \widehat{b} \right\rangle_R) \\
\left\langle \widehat{S}_y \right\rangle_R &= \frac{1}{2i}(\left\langle \widehat{b}^\dagger \right\rangle_R \langle \widehat{a} \rangle_R - \langle \widehat{a}^\dagger \rangle_R \left\langle \widehat{b} \right\rangle_R) \qquad (90)
\end{aligned}$$

is now non-zero, since the previously zero terms have again been retained. Hence

$$\left\langle \widehat{S}_x \right\rangle_R^2 + \left\langle \widehat{S}_y \right\rangle_R^2 = \left\langle \widehat{b}^\dagger \right\rangle_R \left\langle \widehat{b} \right\rangle_R \langle \widehat{a}^\dagger \rangle_R \langle \widehat{a} \rangle_R \tag{91}$$

so that we now have

$$\begin{aligned}
& \left\langle \Delta\widehat{S}_x^2 \right\rangle_R + \left\langle \Delta\widehat{S}_y^2 \right\rangle_R \\
= \; & \frac{1}{2}(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R (\langle \widehat{a}^\dagger\widehat{a} \rangle_R + 1) + \frac{1}{2}(\langle \widehat{a}^\dagger\widehat{a} \rangle_R (\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R) + 1) - \left\langle \widehat{b}^\dagger \right\rangle_R \left\langle \widehat{b} \right\rangle_R \langle \widehat{a} \rangle_R \langle \widehat{a}^\dagger \rangle_R \langle \widehat{a} \rangle_R \\
= \; & \frac{1}{2}\left(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger\widehat{a} \rangle_R\right) + \left(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R (\langle \widehat{a}^\dagger\widehat{a} \rangle_R - |\langle \widehat{a} \rangle_R|^2 |\left\langle \widehat{b}^\dagger \right\rangle_R|^2\right)
\end{aligned} \tag{92}$$

But from the Schwarz inequality - which is based on $\left\langle (\widehat{a}^\dagger - \langle \widehat{a}^\dagger \rangle)(\widehat{a} - \langle \widehat{a} \rangle) \right\rangle \geq 0$ for any state

$$|\langle \widehat{a} \rangle_R|^2 \leq \langle \widehat{a}^\dagger\widehat{a} \rangle_R \qquad |\left\langle \widehat{b} \right\rangle_R|^2 \leq \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R \tag{93}$$

so that

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle_R + \left\langle \Delta\widehat{S}_y^2 \right\rangle_R \geq \frac{1}{2}(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger\widehat{a} \rangle_R) \tag{94}$$

and thus from Eq. (32) it follows that for a general non entangled state

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \sum_R P_R \frac{1}{2} (\langle \widehat{n}_b \rangle_R + \langle \widehat{n}_a \rangle_R) \tag{95}$$

However, half the expectation value of the number operator is

$$\frac{1}{2}\left\langle \widehat{N} \right\rangle = \frac{1}{2}\langle (\widehat{n}_a + \widehat{n}_b) \rangle = \sum_R P_R \frac{1}{2} (\langle \widehat{n}_b \rangle_R + \langle \widehat{n}_a \rangle_R) \tag{96}$$

so for a non-entangled state

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{2}\left\langle \widehat{N} \right\rangle \tag{97}$$

This inequality for non-entangled states is given in [23] (see their Eq. (3)). The above proof was based on *not* invoking the SSR requirements for separable states that we apply in this paper.

4.1.2 Validity of Hillery Test for Local SSR Compliant Non-Entangled States

However, it is interesting that the inequality (97) can be more readily derived from the definition of entangled states used in the present paper - which is based on local particle number SSR compliance for separable states. We would then find that $\left\langle \widehat{S}_x \right\rangle_R = \left\langle \widehat{S}_y \right\rangle_R = 0$ and hence

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle_R + \left\langle \Delta\widehat{S}_y^2 \right\rangle_R = \frac{1}{2}\left(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R + \langle \widehat{a}^\dagger\widehat{a} \rangle_R\right) + \left(\left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R (\langle \widehat{a}^\dagger\widehat{a} \rangle_R\right) \tag{98}$$

instead of Eq.(92). Since the term $\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_R$ ($\langle\widehat{a}^{\dagger}\widehat{a}\rangle_R$ is always positive we find after applying Eq. (32) that

$$\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle \geq \frac{1}{2}\left\langle \widehat{N}\right\rangle \tag{99}$$

which is the same as in Eq. (97). Hence, finding that $\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle < \frac{1}{2}\left\langle \widehat{N}\right\rangle$ would show that the state was entangled, irrespective of whether or not entanglement is defined in terms of non-physical unentangled states.

Thus, the *Hillery spin variance* entanglement test [23] is that *if*

$$\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle < \frac{1}{2}\left\langle \widehat{N}\right\rangle \tag{100}$$

then the state is an entangled state of modes $\widehat{a}$ and $\widehat{b}$. This test is still used in recent papers, for example [24], [25] which deal with the entanglement of sub-systems each consisting of single modes $\widehat{a}$, $\widehat{b}$ for a double well situation (in these papers $\widehat{S}_x \rightarrow \widehat{J}_{AB}^X$, $\widehat{S}_y \rightarrow -\widehat{J}_{AB}^Y$, $\widehat{S}_z \rightarrow -\widehat{J}_{AB}^Z$).

4.1.3 Non-Applicable Entanglement Test Involving $|\left\langle \widehat{S}_z\right\rangle|$

Previously we had found for a general non-entangled state that is based on physically valid density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$

$$\begin{aligned}
\left\langle \Delta\widehat{S}_x^2\right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z\right\rangle| &\geq 0 \\
\left\langle \Delta\widehat{S}_y^2\right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z\right\rangle| &\geq 0
\end{aligned} \tag{101}$$

so that the sum of the variances satisfies the inequality

$$\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle \geq |\left\langle \widehat{S}_z\right\rangle| \tag{102}$$

This is another correct inequality required for a non-entangled state as defined in the present paper. It follows that if only physical states $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ are allowed, the related *entanglement test* involving $\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle$ would be

$$\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle < |\left\langle \widehat{S}_z\right\rangle| \tag{103}$$

For *any* quantum state we have

$$|\left\langle \widehat{S}_z\right\rangle| = \frac{1}{2}|\left(\langle\widehat{n}_b\rangle - \langle\widehat{n}_a\rangle\right)| \leq \frac{1}{2}\left(\langle\widehat{n}_b\rangle + \langle\widehat{n}_a\rangle\right) = \frac{1}{2}\left\langle \widehat{N}\right\rangle \tag{104}$$

which means that it is now required that $\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle$ be less than a quantity that is *smaller* than in the criterion in (97).

However, it should be noted (see (12)) that *all* states, entangled or otherwise, satisfy the inequality

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq |\left\langle \widehat{S}_z \right\rangle| \tag{105}$$

so the inequality in (102) - though true, is of no use in establishing whether a state is entangled in the terms of the meaning of entanglement in the present paper. There are *no* quantum states, entangled or otherwise that satisfy the proposed entanglement test given in Eq. (103). This general result was stated by Hillery et al [23]. To show this we have

$$\begin{aligned} \left\langle \left(\Delta\widehat{S}_x - i\lambda\Delta\widehat{S}_y\right)^\dagger \left(\Delta\widehat{S}_x + i\lambda\Delta\widehat{S}_y\right) \right\rangle &\geq 0 & (106) \\ \left\langle \Delta\widehat{S}_x^2 \right\rangle + \lambda \left\langle \widehat{S}_z \right\rangle + \lambda^2 \left\langle \Delta\widehat{S}_y^2 \right\rangle &\geq 0 & (107) \end{aligned}$$

for all real λ. The condition that this function of λ is never negative gives the Heisenberg uncertainty principle $\left\langle \Delta\widehat{S}_x^2 \right\rangle \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{4}|\left\langle \widehat{S}_z \right\rangle|^2$ and (105) follows from taking $\lambda = 1$ and $\lambda = -1$. Even spin squeezed states with $\left\langle \Delta\widehat{S}_x^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ still have $\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq |\left\langle \widehat{S}_z \right\rangle|$, so it is *never* found that $\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|$ and hence this latter inequality *cannot* used as a test for entanglement.

Fortunately - as we have seen, showing that spin squeezing occurs via *either* $\left\langle \Delta\widehat{S}_x^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ *or* $\left\langle \Delta\widehat{S}_y^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ is sufficient to establish that the state is an entangled state for modes $\widehat{a}, \widehat{b}$, with analogous results if principle spin operators are considered. Applying the Hillery et al entanglement test in Eq. (100) involving $\frac{1}{2}\left\langle \widehat{N} \right\rangle$ is also a valid entanglement test, but is usually *less stringent* than the spin squeezing test involving either $\left\langle \Delta\widehat{S}_x^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ *or* $\left\langle \Delta\widehat{S}_y^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$. For the Hillery et al entanglement test to be satisfied at least one of $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ or $\left\langle \Delta\widehat{S}_y^2 \right\rangle$ is required to be less than $\frac{1}{2}\left\langle \widehat{N} \right\rangle$, whereas for the spin squeezing test to apply at least one of $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ or $\left\langle \Delta\widehat{S}_y^2 \right\rangle$ must be less than $\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$. The quantity $\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ is likely to be smaller than $\frac{1}{2}\left\langle \widehat{N} \right\rangle$ - for example the Bloch vector may lie close to the xy plane, so a greater degree of reduction in spin fluctuation is needed to satisfy the spin squeezing test for entanglement.

However, this is not always the case as the example of the *relative phase state* discussed in SubSection 3.8 shows. The results in the current SubSection can easily be modified to apply to new spin operators $\widehat{J}_x$, $\widehat{J}_y$, $\widehat{J}_z$, with entanglement being considered for new modes $\widehat{c}$ and $\widehat{d}$. The Hillery et al [23] entanglement

test then becomes

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle + \left\langle \Delta\widehat{J}_y^2 \right\rangle < \frac{1}{2}\left\langle \widehat{N} \right\rangle \tag{108}$$

In the case of the relative phase eigenstate we have from Eq. (85) that $\left\langle \Delta\widehat{J}_x^2 \right\rangle + \left\langle \Delta\widehat{J}_y^2 \right\rangle = \frac{1}{12}N^2 + \frac{1}{4} + \frac{1}{8}\ln N \approx \frac{1}{12}N^2$ for large N. This clearly exceeds $\frac{1}{2}\left\langle \widehat{N} \right\rangle = \frac{1}{2}N$, so the Hillery et al [23] test for entanglement fails. On the other hand, as we have seen in SubSection 3.8 $\left\langle \Delta\widehat{J}_y^2 \right\rangle < \frac{1}{2}|\left\langle \widehat{J}_z \right\rangle| \approx \frac{\pi}{16}N$, so the spin squeezing test is satisfied for this entangled state of modes $\widehat{c}$ and $\widehat{d}$.

4.1.4 Hillery Entanglement Test - Multi-Mode Case

It turns out that the Hillery spin variance test can also be applied in multi-mode situations, where the spin operators are defined as in SubSection 2.2. As explained in SubSection 3.3 three cases occur in regard to specifying the sub-systems. For *Case 1*, where there are *two sub-systems* each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$. the Hillery spin variance test as in (100) applies. The proof is set out in Appendix 13, and again does not require the sub-system density operators to be local SSR compliant. Also, for *Case 2* where there are $2n$ subsystems consisting of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$ the Hillery spin variance test as in (100) applies. The proof is set out in Appendix 13, and again does not require the sub-system density operators to be local SSR compliant. However, for *Case 3* where there are n sub-systems consisting of *all* mode pairs $\widehat{a}_i$ and $\widehat{b}_i$ the Hillery spin variance test does *not* apply. The reason is explained in Appendix 13. Basically, it is because specific sub-system density operators $\widehat{\rho}_R^{ab(i)}$ (see (64)) could be *entangled* states of the modes $\widehat{a}_i$ and $\widehat{b}_i$ all of which *do* satisfy the Hillery test involving $\left\langle \widehat{N}_i \right\rangle_R$ for this ith sub-system. If we choose a special separable state of the form (64) with just *one* term (no sum over R), it is easy to see that the Hillery test will be satisfied for the full system. However, the full system state involving these sub-systems is still a *separable* state, showing that satisfying the Hillery spin variance test does not always require the state to be entangled.

4.2 He et al 2012

In two papers dealing with EPR entanglement He et al [26], [24] a *four mode* system associated with a double well potential is considered. In the left well 1 there are two localised modes with annihilation operators $\widehat{a}_1$, $\widehat{b}_1$ and in the right well 2 there are two localised modes with annihilation operators $\widehat{a}_2$, $\widehat{b}_2$. The modes in each well are associated with two different internal states A and B. Note that we use a different notation to [26], [24]. This four mode system provides for the possibility of entanglement of *two sub-systems* each consisting of *pairs of modes*. We can therefore still consider *bipartite* entanglement however. With four modes there are three different choices of such sub-systems but

perhaps the most interesting from the point of view of entanglement of spatially separated modes - and hence implications for EPR entanglement - would be to have the two *left well* modes $\widehat{a}_1$, $\widehat{b}_1$ as sub-system 1 and the two *right well* modes $\widehat{a}_2$, $\widehat{b}_2$ as sub-system 2. This is an example of the general Case 3 considered for multi-modes in SubSection 3.3. Consistent with the requirement that the sub-system density operators $\widehat{\rho}_R^{ab(1)}$, $\widehat{\rho}_R^{ab(2)}$ conform to the symmetrisation principle and the super-selection rule, these density operators will not in general represent separable states for their single mode sub-systems $\widehat{a}_1$, $\widehat{b}_1$ or $\widehat{a}_2$, $\widehat{b}_2$ - and may even be entangled states. As a result when considering *non-entangled* states for the **pair of** sub-systems 1 and 2 we now have

$$\left\langle (\widehat{a}_i^\dagger \widehat{b}_i)^n \right\rangle_{ab(i)} = Tr(\widehat{\rho}_R^{ab(i)} (\widehat{a}_i^\dagger \widehat{b}_i)^n) \neq 0 \qquad i = 1, 2 \tag{109}$$

in general. In this case where the sub-systems are *pairs* of modes the spin squeezing entanglement tests as in Eqs.(59) - (61) for sub-systems consisting of *single* modes cannot be applied, as explained for Case 3 in SubSection 3.3 unless there is only one boson in each sub-system. Nevertheless, there are still tests of bipartite entanglement involving spin operators. We next examine entanglement tests in Refs. [26], [24] to see if any changes occur when we invoke the definition of entanglement based on SSR compliance.

4.2.1 Spin Operator Tests for Entanglement

There are numerous choices for defining spin operators, but the most useful would be the *local spin operators* for each well [24] defined by

$$\begin{aligned} \widehat{S}_x^1 &= (\widehat{b}_1^\dagger \widehat{a}_1 + \widehat{a}_1^\dagger \widehat{b}_1)/2 \qquad \widehat{S}_y^1 = (\widehat{b}_1^\dagger \widehat{a}_1 - \widehat{a}_1^\dagger \widehat{b}_1)/2i \qquad \widehat{S}_z^1 = (\widehat{b}_1^\dagger \widehat{b}_1 - \widehat{a}_1^\dagger \widehat{a}_1)/2 \\ \widehat{S}_x^2 &= (\widehat{b}_2^\dagger \widehat{a}_2 + \widehat{a}_2^\dagger \widehat{b}_2)/2 \qquad \widehat{S}_y^2 = (\widehat{b}_2^\dagger \widehat{a}_2 - \widehat{a}_2^\dagger \widehat{b}_2)/2i \qquad \widehat{S}_z^2 = (\widehat{b}_2^\dagger \widehat{b}_2 - \widehat{a}_2^\dagger \widehat{a}_2)/2 \end{aligned} \tag{110}$$

These satisfy the usual angular momentum commutation rules and those or the different wells commute. The squares of the local vector spin operators are related to the total number operators $\widehat{N}_1 = \widehat{b}_1^\dagger \widehat{b}_1 + \widehat{a}_1^\dagger \widehat{a}_1$ and $\widehat{N}_2 = \widehat{b}_2^\dagger \widehat{b}_2 + \widehat{a}_2^\dagger \widehat{a}_2$ as $\sum\limits_\alpha (\widehat{S}_\alpha^1)^2 = (\ \widehat{N}_1/2)(\widehat{N}_1/2 + 1)$ and $\sum\limits_\alpha (\widehat{S}_\alpha^2)^2 = (\ \widehat{N}_2/2)(\widehat{N}_2/2 + 1)$. The *total spin operators* are

$$\widehat{S}_\alpha = \widehat{S}_\alpha^1 + \widehat{S}_\alpha^2 \qquad \alpha = x, y, z \tag{111}$$

and these satisfy the usual angular momentum commutation rules. Hence there may be cases of spin squeezing, but these do not in general provide entanglement tests.

For the local spin operators we have in general

$$\left\langle \widehat{S}_\alpha^1 \right\rangle_{ab(1)} = Tr(\widehat{\rho}_R^{ab(1)}\, \widehat{S}_\alpha^1) \neq 0 \qquad \left\langle \widehat{S}_\alpha^2 \right\rangle_{ab(2)} = Tr(\widehat{\rho}_R^{ab(2)}\, \widehat{S}_\alpha^2) \neq 0 \qquad \alpha = x, y, z \tag{112}$$

based on (109), and applying (31) we see that *in general* $\left\langle \widehat{S}_\alpha \right\rangle \neq 0$ for separable states. Thus the Bloch vector test for entanglement does not apply.

Furthermore, there is no spin squeezing test either. Following a similar approach as in Section 3 we can obtain the following inequalities for separable states of the sub-systems 1 and 2

$$
\begin{aligned}
& \left\langle \Delta\widehat{S}_x^{\,2} \right\rangle - \frac{1}{2}| \left\langle \widehat{S}_z \right\rangle | \\
\geq \quad & \sum_R P_R \left(\left\langle (\Delta\widehat{S}_x^{\,1})^2 \right\rangle - \frac{1}{2}| \left\langle \widehat{S}_z^{\,1} \right\rangle | \right) + \sum_R P_R \left(\left\langle (\Delta\widehat{S}_x^{\,2})^2 \right\rangle - \frac{1}{2}| \left\langle \widehat{S}_z^{\,2} \right\rangle | \right)
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
& \left\langle \Delta\widehat{S}_y^{\,2} \right\rangle - \frac{1}{2}| \left\langle \widehat{S}_z \right\rangle | \\
\geq \quad & \sum_R P_R \left(\left\langle (\Delta\widehat{S}_y^{\,1})^2 \right\rangle - \frac{1}{2}| \left\langle \widehat{S}_z^{\,1} \right\rangle | \right) + \sum_R P_R \left(\left\langle (\Delta\widehat{S}_y^{\,2})^2 \right\rangle - \frac{1}{2}| \left\langle \widehat{S}_z^{\,2} \right\rangle | \right)
\end{aligned}
\tag{114}
$$

Similar inequalities can be obtained for other pairs of spin operators. In neither case can we state that the right sides are always non-negative, For example, each $\widehat{\rho}_R^{ab(1)}$ may be a spin squeezed state for $\widehat{S}_x^{\,1}$ versus $\widehat{S}_y^{\,1}$ and each $\widehat{\rho}_R^{ab(2)}$ may be a spin squeezed state for $\widehat{S}_x^{\,2}$ versus $\widehat{S}_y^{\,2}$. In this case the right side of the first inequality is a negative quantity, so we cannot conclude that the total $\widehat{S}_x$ is *not* squeezed versus $\widehat{S}_y$ for *all* separable states. As the $\widehat{\rho}_R^{ab(1)}$ and $\widehat{\rho}_R^{ab(2)}$ can be chosen independently we see that separable states for the sub-systems 1 and 2 may *be* spin squeezed, so the presence of spin squeezing in a *total* spin operator is not a test for bipartite entanglement in this four mode system. This does not of course preclude tests for bipartite entanglement involving spin operators, as we will now see.

In SubSection **2.8** of paper **1** it was shown that $| \left\langle \widehat{\Omega}_A^\dagger \widehat{\Omega}_B \right\rangle |^2 \leq \left\langle \widehat{\Omega}_A^\dagger \widehat{\Omega}_A \widehat{\Omega}_B^\dagger \widehat{\Omega}_B \right\rangle$ for a non-entangled state of general sub-systems A *and* B, so with $\widehat{\Omega}_A \rightarrow \widehat{S}_-^1 = \widehat{S}_x^1 - i\widehat{S}_y^1$ and $\widehat{\Omega}_B \rightarrow \widehat{S}_-^2 = \widehat{S}_x^2 - i\widehat{S}_y^2 = (\widehat{S}_+^2)^\dagger$ to give

$$
| \left\langle \widehat{S}_+^1 \widehat{S}_-^2 \right\rangle |^2 \leq \left\langle \widehat{S}_+^1 \widehat{S}_-^1 \widehat{S}_+^2 \widehat{S}_-^2 \right\rangle \tag{115}
$$

for a non-entangled state of sub-systems 1 and 2. For the non-entangled state of these two sub-systems we have

$$
\left\langle \widehat{S}_+^1 \widehat{S}_-^2 \right\rangle = \sum_R P_R \left\langle \widehat{S}_+^1 \right\rangle_{ab(1)}^R \left\langle \widehat{S}_-^2 \right\rangle_{ab(2)}^R \tag{116}
$$

which in general is non-zero from Eq.(112).

Hence a valid *entanglement test* involving *spin operators* for sub-systems 1 and 2 - *each* consisting of *two modes* localised in each well exists, **so if**

$$
| \left\langle \widehat{S}_+^1 \widehat{S}_-^2 \right\rangle |^2 > \left\langle \widehat{S}_+^1 \widehat{S}_-^1 \widehat{S}_+^2 \widehat{S}_-^2 \right\rangle \tag{117}
$$

then the two sub-systems are entangled. A similar conclusion is stated in [24], where the criterion was predicted to be satisfied for four mode two well BEC systems. This test for entanglement involves the local spin operators, though it is not then the same as spin squeezing criteria. It is referred to there as *spin entanglement*. Other similar tests may be obtained via different choices of $\widehat{\Omega}_A$ and $\widehat{\Omega}_B$.

4.3 Raymer et al 2003

In a paper also dealing with bipartite entanglement where the sub-systems each consist of two modes, Raymer et al [27] derive entanglement tests involving spin operators for the sub-systems defined in (110). With Hermitian operators $\widehat{\Omega}_A, \widehat{\Lambda}_A$ and $\widehat{\Omega}_B, \widehat{\Lambda}_B$ for the two sub-systems we consider

$$\widehat{U} = \alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B \qquad \widehat{V} = \alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B \tag{118}$$

where α, β are real. Then with $\widehat{\rho} = \sum_R P_R \widehat{\rho}_R$ and $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$ and using (32) it can first be shown that

$$\left\langle \Delta\widehat{U}^2 \right\rangle \geq \sum_R P_R \left\langle \Delta\widehat{U}_R^2 \right\rangle \qquad \left\langle \Delta\widehat{V}^2 \right\rangle \geq \sum_R P_R \left\langle \Delta\widehat{V}_R^2 \right\rangle \tag{119}$$

where $\Delta\widehat{U}_R = \widehat{U} - \left\langle \widehat{U} \right\rangle_R$, $\Delta\widehat{V}_R = \widehat{V} - \left\langle \widehat{V} \right\rangle_R$ with $\left\langle \widehat{U} \right\rangle_R = Tr(\widehat{U}\,\widehat{\rho}_R)$, $\left\langle \widehat{V} \right\rangle_R = Tr(\widehat{V}\,\widehat{\rho}_R)$.

Substituting for $\widehat{U}$ and $\widehat{V}$ from (118) and using $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$ we can then evaluate the various terms as follows.

$$\begin{aligned}
\left\langle \widehat{U}^2 \right\rangle_R &= \alpha^2 \left\langle \widehat{\Omega}_A^2 \right\rangle_A^R + \beta^2 \left\langle \widehat{\Omega}_B^2 \right\rangle_B^R + 2\alpha\beta \left\langle \widehat{\Omega}_A \right\rangle_A^R \left\langle \widehat{\Omega}_B \right\rangle_B^R \\
\left\langle \widehat{U} \right\rangle_R &= \alpha \left\langle \widehat{\Omega}_A \right\rangle_A^R + \beta \left\langle \widehat{\Omega}_B \right\rangle_B^R \\
\left(\left\langle \widehat{U} \right\rangle_R \right)^2 &= \alpha^2 \left(\left\langle \widehat{\Omega}_A \right\rangle_A^R \right)^2 + \beta^2 \left(\left\langle \widehat{\Omega}_B \right\rangle_B^R \right)^2 + 2\alpha\beta \left\langle \widehat{\Omega}_A \right\rangle_A^R \left\langle \widehat{\Omega}_B \right\rangle_B^R \\
\left\langle \Delta\widehat{U}_R^2 \right\rangle &= \alpha^2 \left(\left\langle \widehat{\Omega}_A^2 \right\rangle_A^R - \left(\left\langle \widehat{\Omega}_A \right\rangle_A^R \right)^2 \right) + \beta^2 \left(\left\langle \widehat{\Omega}_B^2 \right\rangle_B^R - \left(\left\langle \widehat{\Omega}_B \right\rangle_B^R \right)^2 \right)
\end{aligned} \tag{120}$$

with a similar result for $\left\langle \Delta\widehat{V}_R^2 \right\rangle$. Here for sub-system A we define $\left\langle \widehat{\Omega}_A^2 \right\rangle_A^R = Tr(\widehat{\Omega}_A^2\,\widehat{\rho}_R^A)$, $\left\langle \widehat{\Omega}_A \right\rangle_A^R = Tr(\widehat{\Omega}_A\,\widehat{\rho}_R^A)$ and $\left\langle \widehat{\Lambda}_A^2 \right\rangle_A^R = Tr(\widehat{\Lambda}_A^2\,\widehat{\rho}_R^A)$, $\left\langle \widehat{\Lambda}_A \right\rangle_A^R = Tr(\widehat{\Lambda}_A\,\widehat{\rho}_R^A)$ with analogous expressions for sub-system B.

We thus have

$$\begin{aligned}
\left\langle \Delta\widehat{U}^2 \right\rangle &\geq \alpha^2 \sum_R P_R \left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \beta^2 \sum_R P_R \left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R \\
\left\langle \Delta\widehat{V}^2 \right\rangle &\geq \alpha^2 \sum_R P_R \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R + \beta^2 \sum_R P_R \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R
\end{aligned} \tag{121}$$

where $\Delta\widehat{\Omega}_{AR} = \widehat{\Omega}_A - \left\langle \widehat{\Omega}_A \right\rangle_A^R$, $\Delta\widehat{\Omega}_{BR} = \widehat{\Omega}_B - \left\langle \widehat{\Omega}_B \right\rangle_B^R$, $\Delta\widehat{\Lambda}_{AR} = \widehat{\Lambda}_A - \left\langle \widehat{\Lambda}_A \right\rangle_A^R$ and $\Delta\widehat{\Lambda}_{BR} = \widehat{\Lambda}_B - \left\langle \widehat{\Lambda}_B \right\rangle_B^R$.

Adding the two results gives

$$\begin{aligned}
&\left\langle \Delta\widehat{U}^2 \right\rangle + \left\langle \Delta\widehat{V}^2 \right\rangle \\
\geq\ & \alpha^2 \sum_R P_R \left(\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R \right) \\
& + \beta^2 \sum_R P_R \left(\left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R + \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R \right)
\end{aligned} \tag{122}$$

a general variance inequality for separable states.

This last result can be developed further based on the *commutation rules*

$$[\widehat{\Omega}_A, \widehat{\Lambda}_A] = i\widehat{\Theta}_A \qquad [\widehat{\Omega}_B, \widehat{\Lambda}_B] = i\widehat{\Theta}_B \tag{123}$$

The *Schwarz* inequalities - valid for all real λ_A and λ_B

$$\begin{aligned}
\left\langle (\Delta\widehat{\Omega}_{AR} - i\lambda_A \Delta\widehat{\Lambda}_{AR})\, \widehat{\rho}_R^A\, (\Delta\widehat{\Omega}_{AR} + i\lambda_A \Delta\widehat{\Lambda}_{AR}) \right\rangle_A^R &\geq 0 \\
\left\langle (\Delta\widehat{\Omega}_{BR} - i\lambda_B \Delta\widehat{\Lambda}_{BR})\, \widehat{\rho}_R^B\, (\Delta\widehat{\Omega}_{BR} + i\lambda_B \Delta\widehat{\Lambda}_{BR}) \right\rangle_B^R &\geq 0
\end{aligned} \tag{124}$$

lead to the following inequalities

$$\begin{aligned}
\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \lambda_A \left\langle \widehat{\Theta}_A \right\rangle_A^R + \lambda_A^2 \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R &\geq 0 \\
\left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R + \lambda_B \left\langle \widehat{\Theta}_B \right\rangle_B^R + \lambda_B^2 \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R &\geq 0
\end{aligned} \tag{125}$$

so by taking $\lambda_{A,B} = 1$ or -1 we have

$$\begin{aligned}
\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R &\geq |\left\langle \widehat{\Theta}_A \right\rangle_A^R| \\
\left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R + \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R &\geq |\left\langle \widehat{\Theta}_B \right\rangle_B^R|
\end{aligned} \tag{126}$$

The Heisenberg Uncertainty principle results $\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R \geq |\left\langle \widehat{\Theta}_A \right\rangle_A^R|^2/4$ etc also follow from (125).

Noting that $\sum_R P_R \, | \left\langle \widehat{\Theta}_A \right\rangle_A^R | \, \geq | \sum_R P_R \, \left\langle \widehat{\Theta}_A \right\rangle_A^R | \, = | \left\langle \widehat{\Theta}_A \right\rangle |$ and $\sum_R P_R \, | \left\langle \widehat{\Theta}_B \right\rangle_B^R | \, \geq | \sum_R P_R \, \left\langle \widehat{\Theta}_B \right\rangle_A^R | \, = | \left\langle \widehat{\Theta}_B \right\rangle |$ since the modulus of a sum is never greater than the sum of the moduli, we finally arrive at the final inequality for *separable* states

$$\left\langle \Delta(\alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B)^2 \right\rangle + \left\langle \Delta(\alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B)^2 \right\rangle \geq \alpha^2 | \left\langle \widehat{\Theta}_A \right\rangle | \, + \beta^2 | \left\langle \widehat{\Theta}_B \right\rangle | \tag{127}$$

This leads to the following test for *entanglement*

$$\left\langle \Delta(\alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B)^2 \right\rangle + \left\langle \Delta(\alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B)^2 \right\rangle < \alpha^2 | \left\langle \widehat{\Theta}_A \right\rangle | \, + \beta^2 | \left\langle \widehat{\Theta}_B \right\rangle | \tag{128}$$

which is usually based on choices where $\alpha^2 = \beta^2 = 1$.

We now choose $\widehat{\Omega}_A = \widehat{S}\,^1_x$, $\widehat{\Omega}_B = \widehat{S}\,^2_x$, $\widehat{\Lambda}_A = \widehat{S}\,^1_y$ and $\widehat{\Lambda}_B = \widehat{S}\,^2_y$ as in Eq. (110) along with $\alpha = \beta = 1$ Here $\widehat{\Theta}_A = \widehat{S}\,^1_z$ and $\widehat{\Theta}_{\backslash B} = \widehat{S}\,^2_z$.Here sub-systems $A = 1$, $B = 2$ consist of modes $\widehat{a}_1$, $\widehat{b}_1$ and $\widehat{a}_2$, $\widehat{b}_2$ respectively. Hence if we have

$$\left\langle \Delta(\widehat{S}\,^1_x + \widehat{S}\,^2_x)^2 \right\rangle + \left\langle \Delta(\widehat{S}\,^1_y - \widehat{S}\,^2_y)^2 \right\rangle < | \left\langle \widehat{S}^1_z \right\rangle | \, + | \left\langle \widehat{S}^2_z \right\rangle | \tag{129}$$

then *bipartite entanglement* is established. Note that this test did not require local particle number SSR compliance, but still will apply if this is invoked. Other tests involving a cyclic interchange of x, y, z can also be established, as can other tests where the signs within the left terms are replaced by $(-,+), (+,+), (-,-)$ via appropriate choices of α, β. . These tests involve mean values of powers of *local* spin operators. Similar to tests in SubSections 4.1, 4.2, this test also does not require SSR compliance.

4.4 Sorensen et al 2001

4.4.1 Sorensen Spin Squeezing Entanglement Test

In a paper entitled "Many-particle entanglement with Bose-Einstein condensates" Sorensen et al [13] consider the implications for spin squeezing for non-entangled states of the form

$$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}^1_R \otimes \widehat{\rho}^2_R \otimes \widehat{\rho}^3_R \otimes ... \tag{130}$$

where $\widehat{\rho}^i_R$ is a density operator for particle i. As discussed previously, a density operator of this general form is not consistent with the symmetrisation principle - having separate density operators $\widehat{\rho}^i_R$ for specific particles i in an identical particle system (such as for a BEC) is not compatible with the indistinguishability of such particles. It is modes that are distinguishable, not identical particles, so the basis for applying their results to systems of identical bosons is flawed.

However, they derive an inequality for the spin variance $\left\langle \Delta\widehat{S}_z^2 \right\rangle$

$$\left\langle \Delta\widehat{S}_z^2 \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right) \tag{131}$$

that applies in the case of non-entangled states. Key steps in their derivation are stated in the Appendix to [13], but as the justification of these steps is not obvious for completeness the full derivation is given in Appendix 14 of the present paper. This inequality (131) establishes that if

$$\xi^2 = \frac{\left\langle \Delta\widehat{S}_z^2 \right\rangle}{\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right)} < \frac{1}{N} \tag{132}$$

then the state is entangled, so that if the condition for spin squeezing analogous to that in Eq. (16) is satisfied, then entanglement is required if spin squeezing for $\widehat{S}_z$ to occur. Spin squeezing is then a test for entanglement in terms of their definition of an entangled state.

If the Bloch vector is close to the Bloch sphere, for example with $\left\langle \widehat{S}_x \right\rangle = 0$ and $\left\langle \widehat{S}_y \right\rangle = N/2$ then the condition (132) is equivalent to

$$\left\langle \Delta\widehat{S}_z^2 \right\rangle < \frac{1}{2}| \left\langle \widehat{S}_y \right\rangle | \tag{133}$$

which is the condition for squeezing in $\widehat{S}_z$ compared to $\widehat{S}_x$. Spin squeezing is then a test for entanglement in terms of their definition of an entangled state. Note that the condition (133) requires the Bloch vector to be in the xy plane and close to the Bloch sphere of radius $N/2$. By comparison with (16) we see that the *Sorensen spin squeezing* test is that if there is squeezing in $\widehat{S}_z$ with respect to any spin component in the xy plane *and* the Bloch vector is close to the Bloch sphere, then the state is entangled.

As explained above, the proof of Sorensen et al really applies only when the individual spins are *distinguishable*. It is possible however to modify the work of Sorensen et al [13] to apply to a system of identical bosons in accordance with the symmetrization and super-selection rules if the index i is *re-interpreted* as specifying diffferent modes, for example modes localised on *optical lattice* sites $i = 1, 2, .., n$ or distinct free space *momentum states* listed $i = 1, 2, .., n$. On each lattice site or for each momentum state there would be two modes a, b - for example associated with two different *internal states* - so the sub-system density operator $\widehat{\rho}_R^i$ then applies to the two modes on site i. However the proof of (131) requires the $\widehat{\rho}_R^i$ to be restricted to the case where there is exactly *one* identical boson on each site or in a momentum state. Such a localisation process in position or momentum space has the effect of enabling the identical bosons to be treated *as if* they are distinguishable. Details are given in the next SubSections. A similar modification has been carried out by Hyllus et al [29].

However, as we have seen in SubSection 3.2 it does in fact turn out for two mode systems of identical bosons that showing that $\widehat{S}_z$ is spin squeezed compared to $\widehat{S}_x$ or $\widehat{S}_y$ *is* sufficient to prove that the quantum state is entangled. There are *no* restrictions either on the mean number of bosons occupying each mode. The proof is based on applying the requirement of local particle number SSR compliance to the separable states in the present case of massive bosons and treating modes (not particles) as sub-systems. In SubSection 3.3 we have also shown that the same result applies to multi-mode situations in cases where the sub-systems are all single modes (Case2) or where there are two sub-systems each containing all modes for a single component (Case1). So the spin squeezing test is still *valid* for many particle BEC, though the justification is not as in the proof of Sorensen et al [13] - which was derived for systems of distinguishable particles, with each sub-system being a single two state particle.

4.4.2 Revising Sorensen Spin Squeezing Entanglement Test - Localised Modes

The work of Sorensen et al really applies only when the individual spins are distinguishable. It is possible however to modify the work of Sorensen et al [13] to apply to a system of identical bosons in accordance with the symmetrisation and super-selection rules if the index i is *re-interpreted* as specifying diffferent modes, for example modes localised on *optical lattice* sites or in different *momentum states* $i = 1, 2, .., n$. Another example would be single two state ions with each ion being trapped in a different spatial region. The revised approach draws on the results established for multi-mode cases in Appendices 11 and 12. With two single particle states a, b available on each site (these could be two different internal atomic states or two distinct spatial modes localised on the site) the modes would then be labelled $|\phi_{\alpha i}\rangle$ with $\alpha = a, b$. The mode orthogonality and completeness relations would then be

$$\begin{aligned} \left\langle \phi_{\alpha\, i} | \phi_{\beta\, j} \right\rangle &= \delta_{\alpha\beta}\delta_{ij} \\ \sum_{\alpha i} \left| \phi_{\alpha\, i} \right\rangle \left\langle \phi_{\alpha\, i} \right| &= \widehat{1} \end{aligned} \tag{134}$$

With the particles now labelled $K = 1, 2, 3, ...$one can define spin operators in first quantization via

$$\begin{aligned} \widehat{S}_x &= \sum_K \sum_i (\left|\phi_{b\,i}(K)\right\rangle \left\langle \phi_{a\,i}(K)\right| + \left|\phi_{a\,i}(K)\right\rangle \left\langle \phi_{b\,i}(K)\right|)/2 \\ \widehat{S}_y &= \sum_K \sum_i (\left|\phi_{b\,i}(K)\right\rangle \left\langle \phi_{a\,i}(K)\right| - \left|\phi_{a\,i}(K)\right\rangle \left\langle \phi_{b\,i}(K)\right|)/2i \\ \widehat{S}_z &= \sum_K \sum_i (\left|\phi_{b\,i}(K)\right\rangle \left\langle \phi_{b\,i}(K)\right| - \left|\phi_{a\,i}(K)\right\rangle \left\langle \phi_{a\,i}(K)\right|)/2 \end{aligned} \tag{135}$$

In second quantization if the annihilation, creation operators for the modes $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$ are $\widehat{a}_i$, $\widehat{b}_i$ and $\widehat{a}_i^\dagger$, $\widehat{b}_i^\dagger$ respectively, then the Schwinger spin operators are

just

$$\begin{aligned}
\widehat{S}_x &= \sum_i(\widehat{b}_i^\dagger\widehat{a}_i+\widehat{a}_i^\dagger\widehat{b}_i)/2=\sum_i\widehat{S}_x^i \\
\widehat{S}_y &= \sum_i(\widehat{b}_i^\dagger\widehat{a}_i-\widehat{a}_i^\dagger\widehat{b}_i)/2i=\sum_i\widehat{S}_y^i \\
\widehat{S}_z &= \sum_i(\widehat{b}_i^\dagger\widehat{b}_i-\widehat{a}_i^\dagger\widehat{a}_i)/2=\sum_i\widehat{S}_z^i
\end{aligned} \tag{136}$$

It is easy to confirm that the overall spin operators $\widehat{S}_\alpha$ and the spin operators $\widehat{S}_\alpha^i$ for the separate *pairs* of *modes* $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ (or $\widehat{a}_i$, $\widehat{b}_i$ for short) satisfy the same commutation rules as Sorensen et al [13] have for the overall spin operators and those for the separate *particles*. With this modification the non-entangled state in Eq. (130) could be interpreted as being a non-entangled state where the subsystems are actually *pairs* of *modes* $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$ and the density operators $\widehat{\rho}_R^i$ would then refer to a subsystem consisting of these pairs of modes. This corresponds to Case 3 discussed in SubSection 3.3. It is to be noted that entanglement of *pairs* of modes is different to entanglement of *all separate* modes - Case 2 discussed in SubSection 3.3. It is an example of a special kind of *multi-mode entanglement* - since the modes $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$ may themselves be entangled we may have "entanglement of entanglement". In terms of the present paper the density operators $\widehat{\rho}_R^i$ would be restricted by the super-selection rule to statistical mixtures of states with specific total numbers N_i of bosons in the pair of modes $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$. In terms of Fock states $|n_{a\,i}\rangle$, $|n_{b\,i}\rangle$ for this pair of modes the allowed quantum states for the sub-system will be

$$|\Phi_{N_i}\rangle=\sum_{k=0}^{N_i}A_k^{N_i}\;|k\rangle_{a\,i}\,|N_i-k\rangle_{b\,i} \tag{137}$$

so at this stage the general mixed physical state for the two mode system *could* be

$$\widehat{\rho}_R^i=\sum_{N_i=0}^{\infty}\sum_{\Phi}P_{\Phi N_i}\sum_{k=0}^{N_i}\sum_{l=0}^{N_i}A_k^{N_i}\,(A_l^N)^*\,|k\rangle_{a\,i}\,\langle l|_{a\,i}\otimes|N_i-k\rangle_{b\,i}\,\langle N_i-l|_{b\,i} \tag{138}$$

This state has no coherences between states of the two mode subsystem with differing total boson number N_i for the pair of modes. However this is still an entangled states for the two modes $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$, so the overall state in Eq. (138) is not a separable state if the subsystems were to consist of *all* the distinct modes.

4.4.3 Revising Sorensen Spin Squeezing Entanglement Test - Separable State of Single Modes

It is possible however to link spin squeezing and entanglement in the case where the sub-systems consist of *all* the distinct modes (Case2 in SubSection 3.3). To obtain a *fully non-entangled state* of *all* the modes $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$ the density

operator $\widehat{\rho}_R^i$ must then be a product of density operators for modes $|\phi_{ai}\rangle$ and $|\phi_{bi}\rangle$

$$\widehat{\rho}_R^i = \widehat{\rho}_R^{a\,i} \otimes \widehat{\rho}_R^{b\,i} \tag{139}$$

giving the full density operator as

$$\widehat{\rho} = \sum_R P_R \left(\widehat{\rho}_R^{a\,1} \otimes \widehat{\rho}_R^{b\,1}\right) \otimes \left(\widehat{\rho}_R^{a\,2} \otimes \widehat{\rho}_R^{b\,2}\right) \otimes \left(\widehat{\rho}_R^{a\,3} \otimes \widehat{\rho}_R^{b\,3}\right) \otimes . \tag{140}$$

as is required for a general non-entangled state all $2N$ modes. Furthermore, as previously the density operators for the individual modes must represent possible physical states for such modes, so the super-selection rule for atom number will apply and we have

$$\begin{aligned} \langle(\widehat{a}_i)^n\rangle_{a\,i} &= Tr(\widehat{\rho}_R^{a\,i}(\widehat{a}_i)^n) = 0 \qquad \left\langle(\widehat{a}_i^\dagger)^n\right\rangle_{a\,i} = Tr(\widehat{\rho}_R^{a\,i}(\widehat{a}_i^\dagger)^n) = 0 \\ \left\langle(\widehat{b}_i)^m\right\rangle_{b\,i} &= Tr(\widehat{\rho}_R^{b\,i}(\widehat{b}_i)^m) = 0 \qquad \left\langle(\widehat{b}_i^\dagger)^m\right\rangle_{b\,i} = Tr(\widehat{\rho}_R^{b\,i}(\widehat{b}_i^\dagger)^m) = 0 \end{aligned} \tag{141}$$

The question is whether this reformulation will lead to a useful inequality for the spin variances such as $\left\langle \Delta\widehat{S}_x^2 \right\rangle$. This issue is dealt with in Appendix 12 and it is found that we can indeed show for the general *fully non-entangled* state (140) that

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \quad and \quad \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \tag{142}$$

This shows that if there is spin squeezing in *either* $\widehat{S}_x$ *or* $\widehat{S}_y$ then the state must be entangled. Note that this result depends on the general non-entangled state being non-entangled for *all* modes and that the density operator for each mode $\widehat{a}_i$ or $\widehat{b}_i$ being a physical state with no coherences between mode Fock states with differing atom numbers. In terms of the revised interpretation of the density operator to refer to a multi-mode system with modes $|\phi_{ai}\rangle$,$|\phi_{bi}\rangle$ the statement that spin squeezing for systems of identical massive bosons requires all the modes to be entangled is correct. However superposition states of the form (137) that are consistent with the super-selection rule applying to pure states of a two mode system are precluded, and such states ought to be allowed if entanglement of *pairs* of modes rather than of *separate* modes is to be considered.

In addition, we can show that if either $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero, then the state must be entangled - the *Bloch vector* test. Finally, if it is found that if there is spin squeezing in $\widehat{S}_z$ then the state must be entangled. Thus spin squeezing in *any* spin component confirms entanglement of the $2n$ individual modes.

4.4.4 Revising Sorensen Spin Squeezing Entanglement Test - Separable State of Pairs of Modes with One Boson Occupancy

It is also possible however to link spin squeezing and entanglement in the case where the subsystems consist of *pairs* of modes (Case3 in SubSection 3.3), but

only if *further restrictions* are applied. The general *non-entangled* state of the *pairs* of *modes* would actually be of the form (see (64), here the ab dropped for simplicity)

$$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R^1 \otimes \widehat{\rho}_R^2 \otimes \widehat{\rho}_R^3 \otimes ... \tag{143}$$

where the $\widehat{\rho}_R^i$ are now of the form given in Eq. (138) and no longer are density operators for the ith identical particle. Unlike in (141) we now have expectation values $\langle (\widehat{a}_i)^n \rangle_i = Tr(\widehat{\rho}_R^i (\widehat{a}_i)^n)$ etc that are non-zero, so considerations of the link between spin squeezing and entanglement - now entanglement of pairs of modes, willl be different.

If the density operators $\widehat{\rho}_R^i$ associated with the *pair* of modes $\widehat{a}_i$, $\widehat{b}_i$ are all *restricted* to be associated with *one boson states* then this density operator is of the form

$$\begin{aligned} \widehat{\rho}_R^i \;=\; & \rho_{aa}^i (\left|1\right\rangle_{ia} \left\langle 1\right|_{ia} \otimes \left|0\right\rangle_{ib} \left\langle 0\right|_{ib}) + \rho_{ab}^i (\left|1\right\rangle_{ia} \left\langle 0\right|_{ia} \otimes \left|0\right\rangle_{ib} \left\langle 1\right|_{ib}) \\ & + \rho_{ba}^i (\left|0\right\rangle_{ia} \left\langle 1\right|_{ia} \otimes \left|1\right\rangle_{ib} \left\langle 0\right|_{ib}) + \rho_{bb}^i (\left|0\right\rangle_{ia} \left\langle 0\right|_{ia} \otimes \left|1\right\rangle_{ib} \left\langle 1\right|_{ib}) \end{aligned} \tag{144}$$

where the ρ_{ef}^i are density matrix elements. With this restriction the pair of modes $\widehat{a}_i$, $\widehat{b}_i$ behave like *distinguishable* two state particles, essentially the case that Sorensen et al [13] implicitly considered. The expectation values for the spin operators $\widehat{S}_x^i$, $\widehat{S}_y^i$ and $\widehat{S}_z^i$ associated with the ith pair of modes are then

$$\begin{aligned} \left\langle \widehat{S}_x^i \right\rangle_R &= \frac{1}{2}\left(\rho_{ab}^i + \rho_{ba}^i\right) \qquad \left\langle \widehat{S}_y^i \right\rangle_R = \frac{1}{2i}\left(\rho_{ab}^i - \rho_{ba}^i\right) \\ \left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\left(\rho_{bb}^i - \rho_{aa}^i\right) \end{aligned} \tag{145}$$

If in addition Hermitiancy, positivity, unit trace $Tr(\widehat{\rho}_R^i) = 1$ and $Tr(\widehat{\rho}_R^i)^2 \leq 1$ are used (see Appendix 14) then we can show that ρ_{bb}^i and ρ_{aa}^i are real and positive, $\rho_{ab}^i = (\rho_{ba}^i)^*$ and $\rho_{aa}^i \rho_{bb}^i - |\rho_{ab}^i|^2 \geq 0$. The condition $Tr(\widehat{\rho}_R^i) = 1$ leads to $\rho_{aa}^i + \rho_{bb}^i = 1$, from which $Tr(\widehat{\rho}_R^i)^2 \leq 1$ follows using the previous positivity results. These results enable the matrix elements in (144) to be parameterised in the form

$$\begin{aligned} \rho_{aa}^i &= \sin^2 \alpha_i \qquad \rho_{bb}^i = \cos^2 \alpha_i \\ \rho_{ab}^i &= \sqrt{\sin^2 \alpha_i \, \cos^2 \alpha_i}\, \sin^2 \beta_i \, \exp(+i\phi_i) \qquad \rho_{ba}^i = \sqrt{\sin^2 \alpha_i \, \cos^2 \alpha_i}\, \sin^2 \beta_i \, \exp(-i\phi_i) \end{aligned} \tag{146}$$

where α_i, β_i and ϕ_i are real. In terms of these quantities we then have

$$\begin{aligned} \left\langle \widehat{S}_x^i \right\rangle_R &= \frac{1}{2}\sin 2\alpha_i \, \sin^2 \beta_i \, \cos\phi_i \qquad \left\langle \widehat{S}_y^i \right\rangle_R = \frac{1}{2}\sin 2\alpha_i \, \sin^2 \beta_i \, \sin\phi_i \\ \left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\cos 2\alpha_i \end{aligned} \tag{147}$$

and then a key inequality

$$\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 = \frac{1}{4} - \frac{1}{4}\sin^2 2\alpha_i\,(1-\sin^4\beta_i\,) \leq \frac{1}{4} \tag{148}$$

follows. This result depends on the density operators $\widehat{\rho}_R^i$ being for one boson states, as in (144). The same steps as in Sorensen et al [13] (see Appendix 14) leads to the result

$$\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right) \tag{149}$$

for non-entangled *pair* of modes $\widehat{a}_i$, $\widehat{b}_i$. Thus when the interpretation is changed so that are the separate sub-systems are these pairs of modes *and* the sub-systems are in one boson states, it follows that spin squeezing requires entanglement of all the mode pairs.

A similar proof extending the test of Sorensen et al [13] to appply to systems of identical bosons is given by Hyllus et al [29] based on a particle entanglement approach. In their approach bosons in differing external modes (analogous to differing i here) are treated as distinguishable, and the symmetrization principle is ignored for such bosons.

4.5 Benatti et al 2011

In earlier work Toth and Gunhe [18] derived several spin operator based inequalities for separable states for two mode particle systems based on the assumption that the particles were *distinguishable*. As in Eq.(130), the density operator was not required to satisfy the symmetrisation principle. Tests for entanglement involving the mean values and variances for two mode spin operators resulted. Subsequently, Benatti et al [30] considered whether these tests would still apply if the particles were *indistinguishable*. Their work involves considering states with N bosons.

For *separable* states they found (see Eq.(10)) that for three orthogonal spin operators $\widehat{J}_{n1}$, $\widehat{J}_{n2}$ and $\widehat{J}_{n3}$

$$\left\langle \widehat{J}_{n1}^2 \right\rangle + \left\langle \widehat{J}_{n2}^2 \right\rangle + \left\langle \widehat{J}_{n3}^2 \right\rangle \leq \frac{N(N+2)}{4} \tag{150}$$

from which it might be concluded that if the left side exceeded $N(N+2)/4$ then the state must be entangled. However, since $\widehat{J}_{n1}^2 + \widehat{J}_{n1}^2 + \widehat{J}_{n1}^2 = \widehat{S}_x^2 + \widehat{S}_y^2 + \widehat{S}_z^2 = \widehat{N}(\widehat{N}+2)/4$ the left side is always equal to $N(N+2)/4$ for all states with N bosons, so no entanglement test results. This outcome is for similar reasons as for the failed entanglement test $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle + \left\langle \Delta \widehat{S}_y^{\,2} \right\rangle < |\left\langle \widehat{S}_z \right\rangle|$ discussed in SubSection 4.1.

For *separable* states they also found (see Eq.(11)) that for three orthogonal spin operators $\widehat{J}_{n1}$, $\widehat{J}_{n2}$ and $\widehat{J}_{n3}$

$$\left\langle \Delta \widehat{J}_{n1}^2 \right\rangle + \left\langle \Delta \widehat{J}_{n2}^2 \right\rangle + \left\langle \Delta \widehat{J}_{n3}^2 \right\rangle \geq \frac{N}{2} \tag{151}$$

so that if

$$\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle <\frac{N}{2} \tag{152}$$

then the state must be entangled. This test is an extended form of the Hillery spin variance test (100). To prove this result we note that $\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle =\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle +\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle +\left\langle \Delta\widehat{S}_{z}^{2}\right\rangle =.\left\langle \widehat{S}_{x}^{2}\right\rangle +\left\langle \widehat{S}_{y}^{2}\right\rangle +\left\langle \widehat{S}_{z}^{2}\right\rangle -\left\langle \widehat{S}_{x}\right\rangle ^{2}-\left\langle \widehat{S}_{y}\right\rangle ^{2}-\left\langle \widehat{S}_{z}\right\rangle ^{2}=N(N+2)/4-\left\langle \widehat{S}_{x}\right\rangle ^{2}-\left\langle \widehat{S}_{y}\right\rangle ^{2}-\left\langle \widehat{S}_{z}\right\rangle ^{2}$ for all states with N bosons. For separable states we have $\left\langle \widehat{S}_{x}\right\rangle =\left\langle \widehat{S}_{y}\right\rangle =0$ so that $\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle =N(N+2)/4-\left\langle \widehat{S}_{z}\right\rangle ^{2}$. As the eigenvalues for $\widehat{S}_{z}$ lie between $-N/2$ and $+N/2$ we have $\left\langle \widehat{S}_{z}\right\rangle ^{2}\leq N^{2}/4$. Thus $\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle \geq\frac{N}{2}$ as required. The test in (152) is quite useful in that it applies to any three orthogonal spin operators, though it would be harder to satisfy compared to the Hillery spin variance test because of the additional $.\left\langle \Delta\widehat{S}_{z}^{2}\right\rangle$ term.

For *separable* states they also found (see Eq.(13)) that for three orthogonal spin operators $\widehat{J}_{n1}$, $\widehat{J}_{n2}$ and $\widehat{J}_{n3}$

$$(N-1)\left(\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle \right)-\left\langle \widehat{J}_{n3}^{2}\right\rangle \geq\frac{N(N-2)}{4} \tag{153}$$

so that if

$$(N-1)\left(\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle +\left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle \right)-\left\langle \widehat{J}_{n3}^{2}\right\rangle <\frac{N(N-2)}{4} \tag{154}$$

then the state must be entangled. To prove this result for $n1=\overrightarrow{x}$, $n2=\overrightarrow{y}$ and $n3=\overrightarrow{z}$, we use the result (41) for separable states that $\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle +\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle \geq\sum_{R}P_{R}\,\frac{1}{2}(\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R}+\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R})+\sum_{R}P_{R}\,(\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R})=N/2+\sum_{R}P_{R}\,(\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R})$. It is straightforward to show that $\widehat{S}_{z}^{2}=(\widehat{b}^{\dagger}\widehat{b}+\widehat{a}^{\dagger}\widehat{a})^{2}/4-\widehat{b}^{\dagger}\widehat{b}.\widehat{a}^{\dagger}\widehat{a}$, so that $\left\langle \widehat{S}_{z}^{2}\right\rangle =N^{2}/4-\sum_{R}P_{R}\,(\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R})$. Hence for separable states $(N-1)(\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle +\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle)-\left\langle \widehat{S}_{z}^{2}\right\rangle \geq(N-1)N/2+(N-1)\sum_{R}P_{R}\,(\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R})-N^{2}/4+\sum_{R}P_{R}\,(\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R})$. Thus $(N-1)(\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle +\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle)-\left\langle \widehat{S}_{z}^{2}\right\rangle \geq\frac{N(N-2)}{4}+N\sum_{R}P_{R}\,(\langle\widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle _{R})$. As the second term on the right side is always positive the required inequality follows.

Finally, they considered another inequality (see Eq. (12)) found to apply for *separable* states involving *distinguishable* particles in Ref. [18].

$$\left(\left\langle \widehat{J}_{n1}^{2}\right\rangle +\left\langle \widehat{J}_{n2}^{2}\right\rangle \right)-\frac{N}{2}-(N-1)\left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle \leq0 \tag{155}$$

so the question is whether an entanglement test $\left(\left\langle \widehat{J}_{n1}^{2}\right\rangle + \left\langle \widehat{J}_{n2}^{2}\right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle > 0$ applies for the case of indistinguishable particles. For the case where $n1 = \overrightarrow{x}$, $n2 = \overrightarrow{y}$ and $n3 = \overrightarrow{z}$, $\left(\left\langle \widehat{S}_{x}^{2}\right\rangle + \left\langle \widehat{S}_{y}^{2}\right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{S}_{z}^{2}\right\rangle = \left(\left\langle \widehat{S}_{x}^{2}\right\rangle + \left\langle \widehat{S}_{y}^{2}\right\rangle + \left\langle \widehat{S}_{z}^{2}\right\rangle\right) - N/2 - (N)\left\langle \widehat{S}_{z}^{2}\right\rangle + (N-1)\left\langle \widehat{S}_{z}\right\rangle^{2} = N(N+2)/4 - N/2 - N\left(N^{2}/4 - \sum_{R} P_{R}\left(\left\langle \widehat{a}^{\dagger}\widehat{a}\right\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R}\right)\right) + (N-1)\left\langle \widehat{S}_{z}\right\rangle^{2}$. As $\left\langle \widehat{S}_{z}\right\rangle^{2} \leq N^{2}/4$ we see that $\left(\left\langle \widehat{S}_{x}^{2}\right\rangle + \left\langle \widehat{S}_{y}^{2}\right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{S}_{z}^{2}\right\rangle \leq N\sum_{R} P_{R}\left(\left\langle \widehat{a}^{\dagger}\widehat{a}\right\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R}\right)$, which is certainly ≥ 0 and not ≤ 0 as required. However, perhaps an entanglement test such that if is could be shown that

$$\left(\left\langle \widehat{S}_{x}^{2}\right\rangle + \left\langle \widehat{S}_{y}^{2}\right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{S}_{z}^{2}\right\rangle > N\sum_{R} P_{R}\left(\left\langle \widehat{a}^{\dagger}\widehat{a}\right\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R}\right) \quad (156)$$

always applies then it could be included that the state is entangled. Unfortunately the right side could be too large for the left side to always exceed the right side for some separable states. Noting that $\left\langle \widehat{a}^{\dagger}\widehat{a}\right\rangle_{R} + \left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R} = N$ for the N bosons states being considered we find that the right side is maximised when $\left\langle \widehat{a}^{\dagger}\widehat{a}\right\rangle_{R} = \left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R} = N/2$ for all P_{R}, giving a maximum for the right side of $N^{3}/2$ - and this can occur for some separable states. To show that the state is entangled the left side must exceed this value, otherwise the state might be one of the separable states. However, the left side is at most of order N^{2} from the firsr two terms and the negative terms only make the left side smaller. Hence there is no entanglement test of the form (156).$\left\langle \widehat{a}^{\dagger}\widehat{a}\right\rangle_{R} = \left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R}$

Hence Benatti et al [30] have demonstrated two further entanglement tests (152) and (154) for two mode systems of identical particle that involve spin operators. Again, these tests do not involve invoking the local particle number SSR for separable states.

4.6 Sorensen and Molmer 2001

In a paper entitled "Entanglement and Extreme Spin Squeezing" Sorensen and Molmer [31] first consider the limits imposed by the Heisenberg uncertainty principle on the variance $\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle$ considered as a function of $|\left\langle \widehat{S}_{z}\right\rangle|$ for states with N two mode bosons where the spin operators are chosen such that $\left\langle \widehat{S}_{x}\right\rangle = \left\langle \widehat{S}_{y}\right\rangle = 0$. Note that such spin operators can always be chosen so that the Bloch vector does lie along the z axis, even if the spin operators are not principal spin operators. Their treatment is based on combining the result from the Schwarz inequality

$$\left\langle \widehat{S}_{x}^{2}\right\rangle + \left\langle \widehat{S}_{y}^{2}\right\rangle + \left\langle \widehat{S}_{z}\right\rangle^{2} \leq J(J+1) \quad (157)$$

where $J = N/2$, and the Heisenberg uncertainty principle

$$\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle \left\langle \Delta\widehat{S}_y^{\,2} \right\rangle = \xi \frac{1}{4} |\left\langle \widehat{S}_z \right\rangle|^2 \tag{158}$$

where $\xi \geq 1$. In fact two inequalities can be obtained

$$\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle \geq \frac{1}{2}\left\{ \left(J(J+1) - \left\langle \widehat{S}_z \right\rangle^2 \right) - \sqrt{\left(J(J+1) - \left\langle \widehat{S}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{S}_z \right\rangle^2} \right\} \tag{159}$$

$$\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle \leq \frac{1}{2}\left\{ \left(J(J+1) - \left\langle \widehat{S}_z \right\rangle^2 \right) + \sqrt{\left(J(J+1) - \left\langle \widehat{S}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{S}_z \right\rangle^2} \right\} \tag{160}$$

which restricts the region in a $\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle$ versus $|\left\langle \widehat{S}_z \right\rangle|$ plane that applies for states that are consistent with the Heisenberg uncertainty principle. Note that in the first inequality the minimum value for $\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle$ occurs for $\xi = 1$, and in the second inequality the maximum value for $\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle$ also occurs for $\xi = 1$ - the minimum HUP case. The first of these two inequalities is given as Eq. (3) in [31]. For states in which $\widehat{S}_x$ is squeezed relative to $\widehat{S}_y$ the points in the $\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle$ versus $|\left\langle \widehat{S}_z \right\rangle|$ plane must also satisfy

$$\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle \leq \frac{1}{2} |\left\langle \widehat{S}_z \right\rangle| \tag{161}$$

Note that as $\widehat{J}_z$ is a spin angular momentum component we always have $|\left\langle \widehat{S}_z \right\rangle| \leq J$, which places an overall restriction on $|\left\langle \widehat{S}_z \right\rangle|$. However, for $\xi > 1$ there are values of $|\left\langle \widehat{S}_z \right\rangle|$ which are excluded via the Heisenberg uncertainty principle, since the quantity $\left(J(J+1) - \left\langle \widehat{S}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{S}_z \right\rangle^2$ then becomes negative. This effect is seen in Figure 4.

The question is: Is it possible to find values for $\left\langle \Delta\widehat{S}\,_x^2 \right\rangle$ and $|\left\langle \widehat{S}\,_z \right\rangle|$ in which all three inequalities are satisfied? The answer is yes. Results showing the regions in the $\left\langle \Delta\widehat{S}\,_x^2 \right\rangle$ versus $|\left\langle \widehat{S}\,_z \right\rangle|$ plane corresponding to the three inequalities are shown in Figures 4 and 5 for the cases where $J = 1000$ and with $\xi = 1.0$ and $\xi = 10.0$ respectively. The quantities for which the regions are shown are the scaled variance and mean $\left\langle \Delta\widehat{S}\,_x^2 \right\rangle /J$ and $|\left\langle \widehat{S}\,_z \right\rangle|/J$, with $\left\langle \Delta\widehat{S}\,_x^2 \right\rangle$ given as a function of $|\left\langle \widehat{S}\,_z \right\rangle|$ via (159), (160) and (161). The spin squeezing region is always consistent with the second Heisenberg inequality (160) and for large $J = 1000$ there is a large region of overlap with the first inequality (159). For small J and large ξ the region of overlap becomes much smaller, as the result in Figure 6 for $J = 1$ and with $\xi = 10.0$ shows. As the derivation of the Heisenberg principle inequalities is not obvious, this is set out in Appendix.15.

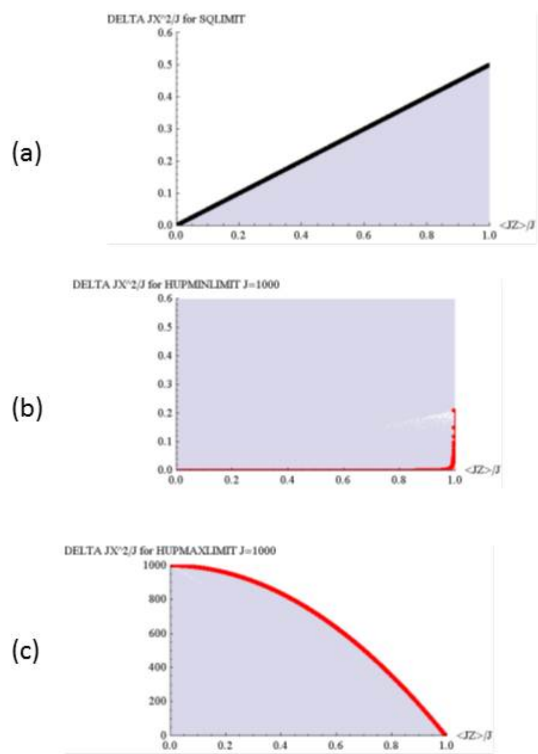

Figure 4. Regions in the $< \Delta\widehat{S}_x^2 >$ versus $| < \widehat{S}_z > |$ plane (shown shaded) for states that satisfy (a) the spin squeezing inequality Eq. (161) (b) the smaller

Heisenberg uncertainty principle inequality Eq. (159) and (c) the larger HUP inequality Eq. (160). The case shown is for $J = 1000$ and HUP factor $\xi = 1$. Both $< \Delta\widehat{S}_x^2 >$ and $| < \widehat{S}_z > |$ are in units of J. The spin operators are chosen so that $< \widehat{S}_x > = < \widehat{S}_y > = 0$.

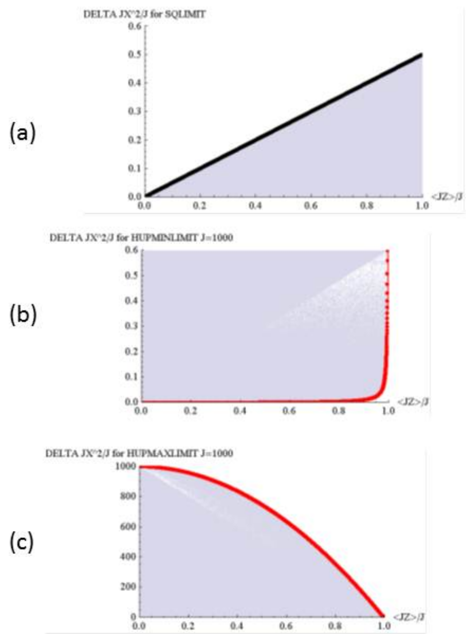

Figure 5. As in Figure 4, but with $J = 1000$ and HUP factor $\xi = 10.0$.

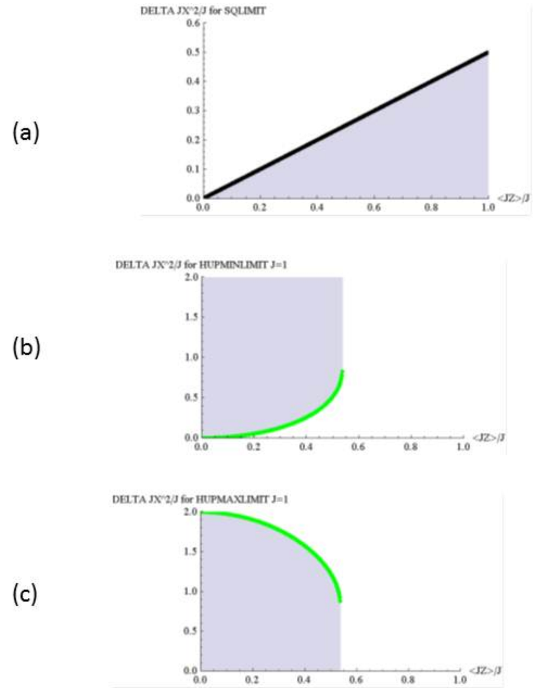

Figure 6. As in Figure 4, but with $J = 1$ and HUP factor $\xi = 10.0$.

Sorensen and Molmer [31] also determine the minimum for $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle = \left\langle \widehat{S}_x^{\,2} \right\rangle$ as a function of $|\left\langle \widehat{S}_z \right\rangle|$ for various choices of J, subject to the constraints $\left\langle \widehat{S}_x \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0$. The results show again that there is a region in the $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle$ versus $|\left\langle \widehat{S}_z \right\rangle|$ plane which is compatible with spin squeezing.

So although these considerations show that the Heisenberg uncertainty principle does not rule out extreme spin squeezing, nothing is yet directly determined about whether the spin squeezed states are entangled states for *modes* $\widehat{a}$, $\widehat{b}$, where the $\widehat{S}_\alpha$ are given as in Eq. (1). The discussion in [31] regarding entanglement is also based on using a density operator for non-entangled states as in Eq. (130) which only applies to distinguishable particles (see SubSection 4.4). Sorensen [31] also showed that for higher J the amount of squeezing attainable could be greater. This fact enables a conclusion to be drawn from the measured spin variance about the minimum number of particles that participate in the non-separable component of an entangled state [32].

5 Correlation Tests for Entanglement

In SubSection **2.4** of the accompanying paper I it was shown that for separable states the inequality $|\left\langle \widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger \right\rangle|^2 \leq \left\langle \widehat{\Omega}_A^\dagger \widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger \widehat{\Omega}_B \right\rangle$ applies, so that if

$$|\left\langle \widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger \right\rangle|^2 > \left\langle \widehat{\Omega}_A^\dagger \widehat{\Omega}_A \otimes \widehat{\Omega}_B^\dagger \widehat{\Omega}_B \right\rangle \tag{162}$$

then the state is entangled. This is a general *correlation* test.

As will be seen the correlation tests can be re-expressed in terms of spin operators when dealing with SSR compliant states.

5.1 Dalton et al 2014

5.1.1 Weak Correlation Test for Local SSR Compliant Non-Entangled States

For a non-entangled state based on *SSR compliant* $\widehat{\rho}_R^A, \widehat{\rho}_R^B$ for modes $\widehat{a}$ and $\widehat{b}$ where the SSR is satisfied we have with $\widehat{\Omega}_A = (\widehat{a})^m$ and $\widehat{\Omega}_B = (\widehat{b})^n$

$$\left\langle (\widehat{a})^m \, (\widehat{b}^\dagger)^n \right\rangle = \sum_R P_R \left\langle (\widehat{a})^m \, (\widehat{b}^\dagger)^n \right\rangle_R = \sum_R P_R \, \langle (\widehat{a})^m \rangle_R \left\langle (\widehat{b}^\dagger)^n \right\rangle_R = 0 \tag{163}$$

since from Eqs. analogous to (28) $\langle (\widehat{a})^m \rangle_R = \left\langle (\widehat{b}^\dagger)^n \right\rangle_R = 0$. Hence for a SSR compliant non-entangled state as defined in the present paper the inequality becomes

$$0 \leq \left\langle (\widehat{a}^\dagger)^m (\widehat{a})^m \, (\widehat{b}^\dagger)^n (\widehat{b})^n \right\rangle \tag{164}$$

which is trivially true and applies for *any* state, entangled or not.

Since $\left\langle (\widehat{a})^m \, (\widehat{b}^\dagger)^n \right\rangle$ is zero for non-entangled states it follows that it is merely necessary to show that this quantity is non-zero to establish that the state is entangled. Hence an *entanglement test* [1] in the case of sub-systems consisting of single modes $\widehat{a}$ and $\widehat{b}$ becomes

$$|\left\langle (\widehat{a})^m \, (\widehat{b}^\dagger)^n \right\rangle|^2 > 0 \tag{165}$$

for a non-entangled state based on *SSR compliant* $\widehat{\rho}_R^A, \widehat{\rho}_R^B$. Note that for globally compliant states $\left\langle (\widehat{a})^m \, (\widehat{b}^\dagger)^n \right\rangle = 0$ unless $n = m$, so only that case is of interest. This is a useful *weak correlation* test for entanglement in terms of the definition of entanglement in the present paper. A related but different test is that of Hillery et al [23] - discussed in the next SubSection.

For the case where $n = m = 1$ the weak correlation test is

$$|\left\langle (\widehat{a}\,\widehat{b}^\dagger \right\rangle|^2 > 0 \tag{166}$$

which is equivalent to $\left\langle \widehat{S}_x \right\rangle \neq 0$ and/or $\left\langle \widehat{S}_y \right\rangle \neq 0$, the Bloch vector test.

5.2 Hillery et al 2006, 2009

5.2.1 Hillery Strong Correlation Entanglement Test

In a later paper entitled "Detecting entanglement with non-Hermitian operators" Hillery et al [33] apply other inequalities for determining entanglement derived in the earlier paper [23] but now also to systems of massive identical bosons, while still retaining .density operators $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ that contain coherences between states with differing boson numbers. In particular, for a non-entangled state the following family of inequalities - originally derived in [23], is invoked.

$$|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n\right\rangle|^2 \leq \left\langle (\widehat{a}^\dagger)^m(\widehat{a})^m\,(\widehat{b}^\dagger)^n(\widehat{b})^n\right\rangle \tag{167}$$

This is just a special case of (162) with $\widehat{\Omega}_A = (\widehat{a})^m$ and $\widehat{\Omega}_B = (\widehat{b})^n$. Thus if

$$|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n\right\rangle|^2 > \left\langle (\widehat{a}^\dagger)^m(\widehat{a})^m\,(\widehat{b}^\dagger)^n(\widehat{b})^n\right\rangle \tag{168}$$

then the state is entangled.The *Hillery* et al [23] entanglement test (168) is a valid test for entanglement and is actually a *more stringent test* than merely showing that $|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n\right\rangle|^2 > 0$, since the quantity $|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n\right\rangle|^2$ is now required to be *larger*. In a paper by He et al [24] (see SubSection 5.3) the Hillery et al [23] entanglement test $|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n\right\rangle|^2 > \left\langle (\widehat{a}^\dagger)^m(\widehat{a})^m\,(\widehat{b}^\dagger)^n(\widehat{b})^n\right\rangle$ is applied for the case where A and B *each* consist of *one mode* localised in each well of a double well potential. This test whilst applicable could be replaced by the more easily satisfied test $|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n\right\rangle|^2 > 0$ (see (165)). However, as will be seen below in SubSection 5.3, the Hillery et al [23] entanglement criterion is needed if the sub-systems each consist of *pairs of modes*, as treated in [26], [24].

Note that if $n \neq m$ the left side is zero for states that are globally SSR compliant. In this case we can always substitute for two mode systems

$$\begin{aligned}
(\widehat{a}\,\widehat{b}^\dagger)^n &= (\widehat{S}_x - i\widehat{S}_y)^n \\
(\widehat{a}^\dagger)^n(\widehat{a})^n &= (\widehat{a}\widehat{a}^\dagger)^n = \left(1 + \frac{\widehat{N}}{2} - \widehat{S}_z\right)^n \\
(\widehat{b}^\dagger)^n(\widehat{b})^n &= (\widehat{b}\widehat{b}^\dagger)^n = \left(1 + \frac{\widehat{N}}{2} + \widehat{S}_z\right)^n
\end{aligned} \tag{169}$$

to write both the Hillery and the weak correlation test in terms of spin operators.

A particular case for $n = m = 1$ is the test $|\left\langle \widehat{a}\,\widehat{b}^\dagger\right\rangle|^2 > \langle \widehat{n}_a\,\widehat{n}_b\rangle$ for an entangled state. To put this result in context, for a general quantum state and any operator $\widehat{\Omega}$ we have $\left\langle \widehat{\Omega}^\dagger\right\rangle = \left\langle \widehat{\Omega}\right\rangle^*$ and $\left\langle \left(\widehat{\Omega}^\dagger - \left\langle \widehat{\Omega}^\dagger\right\rangle\right)\left(\widehat{\Omega} - \left\langle \widehat{\Omega}\right\rangle\right)\right\rangle \geq 0$, hence leading to the Schwarz inequality $|\left\langle \widehat{\Omega}\right\rangle|^2 = |\left\langle \widehat{\Omega}^\dagger\right\rangle|^2 \leq \left\langle \widehat{\Omega}^\dagger\,\widehat{\Omega}\right\rangle$. Taking

$\widehat{\Omega} = \widehat{a}\,\widehat{b}^{\dagger}$ leads to the inequality $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} \leq \langle \widehat{n}_{a}\,(\widehat{n}_{b}+1)\rangle$, whilst choosing $\widehat{\Omega} = \widehat{b}\,\widehat{a}^{\dagger}$ leads to the inequality $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} \leq \langle (\widehat{n}_{a}+1)\,\widehat{n}_{b}\rangle$ for *all* quantum states. In both cases the right side of the inequality is greater than $\langle \widehat{n}_{a}\,\widehat{n}_{b}\rangle$, so *if* it was found that $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} > \langle \widehat{n}_{a}\,\widehat{n}_{b}\rangle$ (though of course still $\leq \langle \widehat{n}_{a}\,(\widehat{n}_{b}+1)\rangle$ and $\leq \langle (\widehat{n}_{a}+1)\,\widehat{n}_{b}\rangle$) then it could be concluded that the state was entangled. However, as we will see the left side $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2}$ actually works out to be zero if physical states for $\widehat{\rho}_{R}^{A}$, $\widehat{\rho}_{R}^{B}$ are involved in defining non-entangled states, so that for a non-entangled state defined as in the present paper the true inequality replacing $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} \leq \langle \widehat{n}_{a}\,\widehat{n}_{b}\rangle$ is just $0 \leq \langle \widehat{n}_{a}\,\widehat{n}_{b}\rangle$, which is trivially true for any quantum state.

For the case where $n = m = 1$ we can write the test (168) in terms of spin operators using $\widehat{a}\,\widehat{b}^{\dagger} = \widehat{S}_{x} - i\widehat{S}_{y}$ as

$$\left\langle \widehat{S}_{x}\right\rangle_{\rho}^{2} + \left\langle \widehat{S}_{y}\right\rangle_{\rho}^{2} > \frac{1}{4}\left\langle \widehat{N}^{2}\right\rangle_{\rho} - \left\langle \widehat{S}_{z}^{2}\right\rangle_{\rho} \tag{170}$$

which when combined with the general result $\widehat{S}_{x}{}^{2}+\widehat{S}_{y}{}^{2}+\widehat{S}_{z}{}^{2} = (\widehat{N}/2)(\widehat{N}/2+1)$ leads to the test

$$\left\langle \Delta\widehat{S}_{x}{}^{2}\right\rangle_{\rho} + \left\langle \Delta\widehat{S}_{y}{}^{2}\right\rangle_{\rho} < \frac{1}{2}\left\langle \widehat{N}\right\rangle_{\rho} \tag{171}$$

This is the same as the Hillery spin variance test (100) , so the Hillery first order correlation test does not add a further test for demonstrating non-SSR compliant entanglement. The Hillery correlation test for $n = 2$ leads to complex conditions involving higher powers of spin operators.

5.2.2 Applications of Correlation Tests for Entanglement

As an example of applying these tests consider the *mixed two mode coherent states* described in Appendix 16, whose density operator for the two mode $\widehat{a}$, $\widehat{b}$ system is given in Eq. (336). We can now examine the Hillery et al [33] entanglement test in Eq.(168) and the entanglement test in Eq.(165) for the case where $m = n = 1$. It is straight-forward to show that

$$\begin{aligned}
|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} &= |\alpha|^{4} \\
\left\langle (\widehat{a}^{\dagger}\widehat{a})\,(\widehat{b}^{\dagger}\widehat{b})\right\rangle &= |\alpha|^{4}
\end{aligned} \tag{172}$$

so that $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} = \left\langle (\widehat{a}^{\dagger}\widehat{a})\,(\widehat{b}^{\dagger}\widehat{b})\right\rangle$. A non-entangled state defined in terms of the SSR requirement for the separate modes satisfies $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} = 0$, whilst for a non-entangled state in which the SSR requirement for separate modes is not specifically required merely satisfies $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^{2} \leq \left\langle (\widehat{a}^{\dagger}\widehat{a})\,(\widehat{b}^{\dagger}\widehat{b})\right\rangle$. Hence the test

for entanglement of modes A, B in the present paper $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^2 > 0$ is satisfied, whilst the Hillery et al [33] test $|\left\langle \widehat{a}\,\widehat{b}^{\dagger}\right\rangle|^2 > \left\langle (\widehat{a}^{\dagger}\widehat{a})\,(\widehat{b}^{\dagger}\widehat{b})\right\rangle$ is not.

In terms of the definition of non-entangled states in the present paper, the mixture of two mode coherent states given in Eq.(336) is *not* a *separable* state, not a separable state. As discussed in Paper 1 (see **Section 3.4.3**) this is because a coherent state gives rise to a non-zero coherence ($\langle\widehat{a}\rangle \neq 0$) and thus cannot represent a physical state for the SSR compliant states involving identical massive bosons (as in BECs). However, in terms of the definition of non-entangled states in other papers such as those of Hillery et al [23], [33] the mixture of two mode coherent states would be a *non-entangled* state. It is thus a useful state for providing an example of the different outcomes of definitions where the local SSR is applied or not.

A further example of applying correlation tests is provided by the *NOON state* defined in (75) where here we consider modes A, B. All matrix elements of the form $\left\langle (\widehat{a}^{\dagger})^m(\widehat{a})^m\,(\widehat{b}^{\dagger})^n(\widehat{b})^n\right\rangle$ are zero for all m, n because both terms contain one mode with zero bosons. Matrix elements of the form $\left\langle (\widehat{a})^m\,(\widehat{b}^{\dagger})^n\right\rangle$ are all zero unless $m = n = N$ and in this case

$$\begin{aligned}
\left\langle (\widehat{a})^N\,(\widehat{b}^{\dagger})^N\right\rangle &= \left\langle (\widehat{S}_x - i\widehat{S}_y)^N\right\rangle \\
&= \cos\theta\,\sin\theta\,\langle N,0|\,(\widehat{S}_-)^N\,|0,N\rangle \\
&= \cos\theta\,\sin\theta\,\sqrt{N}\sqrt{2N-2}\sqrt{3N-6}\sqrt{4N-12}...\sqrt{N} \qquad (173)
\end{aligned}$$

which is non-zero in general. Hence $|\left\langle (\widehat{a})^N\,(\widehat{b}^{\dagger})^N\right\rangle|^2 > 0$ as required for both the weak and strong correlation tests, confirming that the NOON state is *entangled*. Carrying out this entanglement test experimentally for large N would involve measuring expectation values of high powers of the spin operators $\widehat{S}_x$ and $\widehat{S}_y$, which is difficult at present.

5.3 He et al 2012

For the *four mode* system associated with a double well described in SubSection 4.2 (see [24]), the inequalities derived by Hillery et al [33] (see SubSection 5.2)

$$|\left\langle (\widehat{a}_i)^m\,(\widehat{b}_j^{\dagger})^n\right\rangle|^2 \leq \left\langle (\widehat{a}_i^{\dagger})^m(\widehat{a}_i)^m\,(\widehat{b}_j^{\dagger})^n(\widehat{b}_j)^n\right\rangle \qquad (174)$$

that apply for two non-entangled sub-systems A and B can now be usefully applied, since in this case the quantities $\left\langle (\widehat{a}_i)^m\,(\widehat{b}_j^{\dagger})^n\right\rangle$ are in general no longer zero for separable states. Thus there is an *entanglement test* for two sub-systems consisting of *pairs of modes.* If

$$\begin{aligned}
|\left\langle (\widehat{a}_i)^m\,(\widehat{b}_j^{\dagger})^n\right\rangle|^2 &> \left\langle (\widehat{a}_i^{\dagger})^m(\widehat{a}_i)^m\,(\widehat{b}_j^{\dagger})^n(\widehat{b}_j)^n\right\rangle \\
for\ any\ of\ i,j &= 1,2 \qquad (175)
\end{aligned}$$

then the quantum state for two sub-systems A and B - *each* consisting of *two modes* localised in each well - is entangled. Again only the case where $m = n$ is relevant for states that are global SSR compliant.

6 Quadrature Tests for Entanglement

In this Section we discuss tests for two mode entanglement involving so called *quadrature* operators - *position* and *momentum* being particular examples of such operators. These tests are distinct from those involving *spin* operators or *correlation* tests - the latter have been shown to be closely related to spin operator tests. The issue of *measurement* of the quadrature variances involved in these tests for the case of two mode systems involving identical massive bosons will be briefly discussed in Section 7. Again we have a situation where tests derived in which local particle number SSR compliance for separable states is ignored are still valid when it is taken into account. However, when local particle number SSR compliance for separable states is actually included new entanglement tests arise. The *two mode quadrature squeezing* test in (203) is an example, though this test is not very useful as it could be replaced by the *Bloch vector* test. The quadrature correlation test in (197) also applies and is equivalent to the *Bloch vector* test. However the non-existent *quadrature variance* test in (185) is an example where there is no generalisation of the previous entanglement test (see (177)) that applied when the SSR were irrelevant.

6.1 Duan et al 2000

6.1.1 Two Distinguishable Particles

A further inequality aimed at providing a signature for entanglement is set out in the papers by Duan et al [28], Toth et al [34]. Duan et al [28] considered a general situation where the system consisted of two disinguishable sub-systems A and B, for which *position* and *momentum* Hermitian operators $\widehat{x}_A, \widehat{p}_A$ and $\widehat{x}_B, \widehat{p}_B$ were involved that satisfied the standard commutation rules $[\widehat{x}_A, \widehat{p}_A] = [\widehat{x}_B, \widehat{p}_B] = i$ in units where $\hbar = 1$. These sub-systems were quite general and could be two *distinguishable* quantum particles A and B, but other situations can also be treated. An inequality was obtained for a two sub-system non-entangled state involving the variances for the commuting observables $\widehat{x}_A + \widehat{x}_B$ and $\widehat{p}_A - \widehat{p}_B$

$$\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle \geq 2 \tag{176}$$

which could be used to establish a *variance test* for entangled states of the A and B sub-systems, so that if

$$\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle < 2 \tag{177}$$

then the sub-systems are entangled. For the case of distinguishable particles such states are possible - consider for example any simultaneous eigenstate of the commuting observables $\widehat{x}_A + \widehat{x}_B$ and $\widehat{p}_A - \widehat{p}_B$. For such a state $\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle$

and $\langle\Delta(\widehat{p}_A-\widehat{p}_B)^2\rangle$ are both zero, so the simultaneous eigenstates are entangled states of *particles* A, B. For simplicity we only set out the case for which $a=1$ in [28]. The proof given in [28] considered separable states of the general form as in Eq.(27) for two sub-systems but where $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are possible states for sub-systems A, B. Consequently, a first quantization case involving *one particle states* could be involved, where super-selection rules were *not* relevant. As explained in the Introduction, the two distinguishable quantum particles are each equivalent to a whole set of single particle states (momentum eigenstates, harmonic oscillator states, ..) that each quantum particle can occupy, and because both $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ represent states for one particle we have $[\widehat{n}_A,\widehat{\rho}_R^A]=[\widehat{n}_B,\widehat{\rho}_R^B]=0$. Because $\widehat{\rho}$ represent a state for the two particles $[\widehat{n}_A+\widehat{n}_B,\widehat{\rho}]=0$, the SSR are still true, though irrelevant in the case of distinguishable quantum particles A and B.

Another inequality that can be established is

$$\langle\Delta(\widehat{x}_A-\widehat{x}_B)^2\rangle+\langle\Delta(\widehat{p}_A+\widehat{p}_B)^2\rangle\geq 2 \tag{178}$$

which could also be used to establish a variance test for entangled states of the A and B sub-systems, so that if

$$\langle\Delta(\widehat{x}_A-\widehat{x}_B)^2\rangle+\langle\Delta(\widehat{p}_A+\widehat{p}_B)^2\rangle<2 \tag{179}$$

then the sub-systems are entangled.

6.1.2 Two Mode Systems of Identical Bosons

However, we can also consider cases of systems of *identical* bosons with two *modes* A, B rather than two *distinguishable* quantum particles A and B. In this case both the sub-systems may involve arbitrary numbers of particles, so it is of interest to see what implications follow from the physical sub-system states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ now being required to satisfy the local particle number SSR, and all quantum states $\widehat{\rho}$ satisfying the global particle number SSR. It is well-known that in two mode boson systems *quadrature operators* can be defined via

$$\begin{aligned}\widehat{x}_A &= \frac{1}{\sqrt{2}}(\widehat{a}+\widehat{a}^\dagger) \qquad \widehat{p}_A=\frac{1}{\sqrt{2}i}(\widehat{a}-\widehat{a}^\dagger)\\ \widehat{x}_B &= \frac{1}{\sqrt{2}}(\widehat{b}+\widehat{b}^\dagger) \qquad \widehat{p}_B=\frac{1}{\sqrt{2}i}(\widehat{b}-\widehat{b}^\dagger)\end{aligned} \tag{180}$$

which have the same commutation rules as the position and momentum operators for distinguishable particles. Thus $[\widehat{x}_A,\widehat{p}_A]=[\widehat{x}_B,\widehat{p}_B]=i$ as for cases where A, B were distinguishable particles.

Since the proof of Eq. (176) in [28] did not involve invoking the SSR, then *if* the inequality in Eq.(177) is satisfied, then the state *would be* an entangled state for *modes* A, B. as well as for distinguishable particles A, B. The situation would *then* be similar to that for the Hillery et al [23], [33] tests - the SSR compliant sub-system states are just a particular case of the set of all sub-system states. However, in regard to spin squeezing and correlation tests for

entanglement, new tests were found when the SSR were explicitly considered and it is *possible* that this could occur here. This turns out not to be the case.

As we will see, the inequality (176) is replaced by an *equation* that is satisfied by *all* quantum states for two mode systems of identical bosons where the global particle number SSR applies. This equation is the same irrespective of whether the state is separable or entangled. To see this we evaluate $\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle$ for states that are global SSR compliant.

Firstly,

$$\langle (\widehat{x}_A + \widehat{x}_B) \rangle = \langle (\widehat{p}_A - \widehat{p}_B) \rangle = 0 \tag{181}$$

since $\langle \widehat{a} \rangle = \left\langle \widehat{b} \right\rangle = \langle \widehat{a}^\dagger \rangle = \left\langle \widehat{b}^\dagger \right\rangle = 0$ for SSR compliant states.

Secondly,

$$\left\langle (\widehat{x}_A + \widehat{x}_B)^2 \right\rangle = \frac{1}{2} \left(\begin{array}{c} \langle \widehat{a}\,\widehat{a}^\dagger \rangle + \left\langle \widehat{a}\,\widehat{b}^\dagger \right\rangle + \left\langle \widehat{b}\,\widehat{a}^\dagger \right\rangle + \left\langle \widehat{b}\,\widehat{b}^\dagger \right\rangle \\ + \langle \widehat{a}^\dagger\,\widehat{a} \rangle + \left\langle \widehat{a}^\dagger\,\widehat{b} \right\rangle + \left\langle \widehat{b}^\dagger\,\widehat{a} \right\rangle + \left\langle \widehat{b}^\dagger\,\widehat{b} \right\rangle \end{array} \right)$$

using $\langle \widehat{a}^2 \rangle = \langle (\widehat{a}^\dagger)^2 \rangle = \left\langle \widehat{b}^2 \right\rangle = \left\langle (\widehat{b}^\dagger)^2 \right\rangle = \left\langle \widehat{a}\,\widehat{b} \right\rangle = \left\langle \widehat{a}^\dagger\,\widehat{b}^\dagger \right\rangle = 0$ for global SSR compliant states. Hence using the commutation rules, introducing the number operator $\widehat{N}$ and the spin operator $\widehat{S}_x$ and using (181) we find that

$$\begin{aligned} \left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle &= \left\langle (\widehat{x}_A + \widehat{x}_B)^2 \right\rangle \\ &= 1 + \langle \widehat{a}^\dagger\,\widehat{a} \rangle + \left\langle \widehat{b}^\dagger\,\widehat{b} \right\rangle + \left\langle \widehat{b}^\dagger\,\widehat{a} \right\rangle + \left\langle \widehat{a}^\dagger\,\widehat{b} \right\rangle \\ &= 1 + \left\langle \widehat{N} \right\rangle + 2 \left\langle \widehat{S}_x \right\rangle \end{aligned} \tag{182}$$

Similarly

$$\left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle = 1 + \left\langle \widehat{N} \right\rangle - 2 \left\langle \widehat{S}_x \right\rangle \tag{183}$$

Thus we have for all globally SSR compliant states

$$\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle = 2 + 2 \left\langle \widehat{N} \right\rangle \tag{184}$$

Since $\left\langle \widehat{N} \right\rangle \geq 0$ for all quantum states we see that the Duan et al inequality (176) for separable states is still satisfied, but because (184) applies for all states irrespective of whether or not they are separable, we see that there is *no* quadrature variance entanglement test of the form

$$\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle < 2 + 2 \left\langle \widehat{N} \right\rangle \tag{185}$$

for the case of two mode systems of identical massive bosons. The situation is similar to the non-existent test $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle + \left\langle \Delta \widehat{S}_y^{\,2} \right\rangle < | \left\langle \widehat{S}_z \right\rangle |$ in Section 4.1.3. The situation contrasts that in Section 4.3, where a test $\left\langle \Delta(\widehat{S}_x^{\,1} + \widehat{S}_x^{\,2})^2 \right\rangle +$

$\left\langle \Delta(\widehat{S}_y^1 - \widehat{S}_y^2)^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|$ establishes entanglement between two sub-systems (1 and 2) - but in this case each consisting of two modes.

We can also show for all globally SSR compliant states that

$$\begin{aligned} \left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle &= 1 + \left\langle \widehat{N} \right\rangle - 2\left\langle \widehat{S}_x \right\rangle & (186) \\ \left\langle \Delta(\widehat{p}_A + \widehat{p}_B)^2 \right\rangle &= 1 + \left\langle \widehat{N} \right\rangle + 2\left\langle \widehat{S}_x \right\rangle & (187) \end{aligned}$$

and hence

$$\left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A + \widehat{p}_B)^2 \right\rangle = 2 + 2\left\langle \widehat{N} \right\rangle \tag{188}$$

but again no entanglement test results.

The universal result (184) for the quadrature variance sum may seem paradoxical in view of the operators $(\widehat{x}_A + \widehat{x}_B)$ and $(\widehat{p}_A - \widehat{p}_B)$ commuting. Mathematically, this would imply that they would then have a complete set of simultaneous eigenvectors $|X_{A,B},\, P_{A,B}\rangle$ such that $(\widehat{x}_A + \widehat{x}_B)\, |X_{A,B},\, P_{A,B}\rangle = X_{A,B}\, |X_{A,B},\, P_{A,B}\rangle$ and $(\widehat{p}_A - \widehat{p}_B)\, |X_{A,B},\, P_{A,B}\rangle = P_{A,B}\, |X_{A,B},\, P_{A,B}\rangle$. For these eigenstates $\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle = \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle = 0$ which contradicts (184) for such states. However, no such eigenstates exist that are globally SSR compliant. For SSR compliant states $|X_{A,B},\, P_{A,B}\rangle$ must be an eigenstate of $\widehat{N}$ and for eigenvalue N we see that $(\widehat{x}_A + \widehat{x}_B)\, |X_{A,B},\, P_{A,B}\rangle_N$ is a linear combination of eigenstates of $\widehat{N}$ with eigenvalues $N \pm 1$. Hence $(\widehat{x}_A + \widehat{x}_B)$ $|X_{A,B},\, P_{A,B}\rangle_N \neq X_{A,B}\, |X_{A,B},\, P_{A,B}\rangle_N$ so simultaneous eigenstates that are SSR compliant do not exist and there is no paradox. As pointed out above, this issue does not arise for the case of two distinguishable particles where the operators $\widehat{x}_A, \widehat{x}_B, \widehat{p}_A$ and $\widehat{p}_B$ are *not* related to mode annihilation and creation operators - as in the present case.

We can also derive *inequalities* for *separable* states involving $\widehat{x}_A, \widehat{p}_A$ and $\widehat{x}_B, \widehat{p}_B$ based on the approach in Section 4.3. Starting with Eq. (127) we choose $\widehat{\Omega}_A = \widehat{x}_A$, $\widehat{\Omega}_B = \widehat{x}_B$, $\widehat{\Lambda}_A = \widehat{p}_A$ and $\widehat{\Lambda}_B = \widehat{p}_B$. Here $\widehat{\Theta}_A = \widehat{1}_A$ and $\widehat{\Theta}_{\backslash B} = \widehat{1}_B$ For *separable* states we have from (127)

$$\left\langle \Delta(\alpha\widehat{x}_A + \beta\widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\alpha\widehat{p}_A - \beta\widehat{p}_B)^2 \right\rangle \geq \alpha^2 + \beta^2 \tag{189}$$

With the choice of $\alpha^2 = \beta^2 = 1$ we then find the following inequalities for separable states

$$\begin{aligned} \left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle &\geq 2 \\ \left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A + \widehat{p}_B)^2 \right\rangle &\geq 2 \\ \left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A + \widehat{p}_B)^2 \right\rangle &\geq 2 \\ \left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle &\geq 2 \end{aligned} \tag{190}$$

depending on the choice of α and β. With $\alpha = \beta = 1$ the first result is obtained and is the same as in (176). This result is consistent with (184). However using

(182), (187), (186) and (183) we have for *global SSR compliant* states - separable *and* non-separable

$$\begin{aligned}
\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle &= 2 + 2\left\langle \widehat{N} \right\rangle \\
\left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A + \widehat{p}_B)^2 \right\rangle &= 2 + 2\left\langle \widehat{N} \right\rangle \\
\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A + \widehat{p}_B)^2 \right\rangle &= 2 + 2\left\langle \widehat{N} \right\rangle + 4\left\langle \widehat{S}_x \right\rangle \\
\left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle &= 2 + 2\left\langle \widehat{N} \right\rangle - 4\left\langle \widehat{S}_x \right\rangle \qquad (191)
\end{aligned}$$

The implications for the first two equalities have been discussed above. In the case of the $(+,+)$ and $(-,-)$ cases, we note that for states with eigenvalue N for $\widehat{N}$ the eigenvalues for $\widehat{S}_x$ lie in the range $-N/2$ to $+N/2$ and hence $\left\langle \widehat{N} \right\rangle \pm 2\left\langle \widehat{S}_x \right\rangle$ is always ≥ 0. Thus (190) will apply for both separable *and* entangled states. Hence for global SSR compliant states none of (190) lead to an entanglement test.

6.1.3 Non SSR Compliant States

On the other hand if *neither* the sub-system *nor* the overall system states are *required* to be SSR compliant - though they may be - we find that for *separable* states

$$\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle_\rho + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle_\rho \geq 2 + 2\left\langle \widehat{N} \right\rangle_\rho + 2(\left\langle \widehat{a}\,\widehat{b} \right\rangle + \left\langle \widehat{a}^\dagger\,\widehat{b}^\dagger \right\rangle) - 2|\left\langle \widehat{a} \right\rangle_\rho + \left\langle \widehat{b}^\dagger \right\rangle_\rho|^2 \qquad (192)$$

so entanglement based on ignoring local particle number SSR in the separable states is now shown if

$$\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle_\rho + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle_\rho < 2 + 2\left\langle \widehat{N} \right\rangle_\rho + 2(\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho + \left\langle \widehat{a}^\dagger\,\widehat{b}^\dagger \right\rangle_\rho) - 2|\left\langle \widehat{a} \right\rangle_\rho + \left\langle \widehat{b}^\dagger \right\rangle_\rho|^2 \qquad (193)$$

However, even if local particle number SSR compliance is ignored for the sub-system states (as in Ref [3]), global particle number SSR compliance is still required for the overall quantum state. This applies to both the separable states and to states that are being tested for entanglement. In this case the quantities $\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho$, $\left\langle \widehat{a}^\dagger\,\widehat{b}^\dagger \right\rangle_\rho$, $\left\langle \widehat{a} \right\rangle_\rho$, $\left\langle \widehat{a}^\dagger \right\rangle_\rho$, $\left\langle \widehat{b} \right\rangle_\rho$ and $\left\langle \widehat{b}^\dagger \right\rangle_\rho$ are all zero, so the entanglement test in (193) would become the same as the *hypothetical* entanglement test (185).

For the sceptic (see Section 3.1.6) who wishes to completely disregard the SSR (both locally and globally) and proposes to use tests based on quadrature variances such as (193) to establish entanglement, the challenge will be to find a way of measuring the allegedly non-zero quantities $\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho$.. $\left\langle \widehat{b}^\dagger \right\rangle_\rho$. This would require some sort of system with a well-defined *phase* reference. Such a measurement is not possible with the beam splitter interferometer discussed in this paper, and the lack of such a detector system would preclude establishing SSR neglected entanglement for systems of identical bosons. Essentially the

same problem arises in testing whether states that are non-SSR compliant exist in single mode systems of massive bosons.

As mentioned previously, the result in Eq. (176) was established in Ref. [28] *without* requiring the sub-system states $\widehat{\rho}_R^A$, $\widehat{\rho}_R^B$ to be compliant with the local particle number SSR or the density operator $\widehat{\rho}$ for the state being tested to comply with the global particle number SSR, as would be the case for physical sub-system and system states of identical bosons. However, in Ref. [28] it was pointed out that *two mode squeezed vacuum* states of the form $|\Phi\rangle = \exp(-r(\widehat{a}^\dagger\widehat{b}^\dagger - \widehat{a}\widehat{b}))\,|0\rangle$ satisfy the entanglement test. However, such stand alone two-mode states are *not* allowed quantum states for massive identical boson systems. as they are not compliant with the global particle number SSR. To create states with correlated pairs of bosons in modes a and b processes such as the *dissociation* of a bosonic *molecular BEC* in a mode M into *pair* of bosonic atoms in modes a and b can indeed occur, but would involve interaction Hamiltonians such as $\widehat{V} = \kappa(\widehat{a}^\dagger\widehat{b}^\dagger\widehat{M} + \widehat{M}^\dagger\widehat{a}\widehat{b})$. The state produced would be an entangled state of the atoms plus molecules which would be compliant with the global total boson number SSR - taking into account the boson particle content of the molecule via $\widehat{N} = 2\widehat{n}_M + \widehat{n}_a + \widehat{n}_b$. It would not be a state of the form $|\Phi\rangle = \exp(-r(\widehat{a}^\dagger\widehat{b}^\dagger - \widehat{a}\widehat{b}))\,|0\rangle$.

6.2 Reid 1989

Another test involves the *general quadrature operators* defined as in [35], for which those in (180) are special cases

$$\begin{aligned}\widehat{X}_a^\theta &= \frac{1}{\sqrt{2}}(\widehat{a}\exp(-i\theta) + \widehat{a}^\dagger\exp(+i\theta)) \\ \widehat{X}_b^\phi &= \frac{1}{\sqrt{2}}(\widehat{b}\exp(-i\phi) + \widehat{b}^\dagger\exp(+i\phi)) \end{aligned} \tag{194}$$

These operators are Hermitian. The conjugate operators are

$$\begin{aligned}\widehat{P}_a^\theta &= \frac{1}{\sqrt{2}i}(\widehat{a}\exp(-i\theta) - \widehat{a}^\dagger\exp(+i\theta)) = \widehat{X}_a^{\theta+\pi/2} \\ \widehat{P}_b^\phi &= \frac{1}{\sqrt{2}i}(\widehat{b}\exp(-i\phi) - \widehat{b}^\dagger\exp(+i\phi)) = \widehat{X}_b^{\phi+\pi/2} \end{aligned} \tag{195}$$

where $[\widehat{X}_a^\theta, \widehat{P}_a^\theta] = [\widehat{X}_b^\phi, \widehat{P}_b^\phi] = i$.

Noting that for any state we have $\left\langle (\widehat{X}_a^\theta - \lambda\widehat{X}_b^\phi)^2 \right\rangle \geq 0$ for all real λ establishes the Cauchy inequality for all quantum states

$$C_{ab}^{\theta\phi} = \frac{|\left\langle \widehat{X}_a^\theta\widehat{X}_b^\phi \right\rangle|^2}{\left\langle (\widehat{X}_a^\theta)^2 \right\rangle \left\langle (\widehat{X}_b^\phi)^2 \right\rangle} \leq 1 \tag{196}$$

The quantity $C_{ab}^{\theta\phi}$ is a *correlation coefficient*. For SSR compliant separable states $\left\langle \widehat{X}_a^\theta\widehat{X}_b^\phi \right\rangle = \sum_R P_R \left\langle \widehat{X}_a^\theta \right\rangle_R \left\langle \widehat{X}_b^\phi \right\rangle_R = 0$, whilst for all globally SSR compliant

states $\left\langle(\widehat{X}_a^\theta)^2\right\rangle = \langle\widehat{n}_a\rangle + \frac{1}{2} > \frac{1}{2}$ and $\left\langle(\widehat{X}_b^\phi)^2\right\rangle = \langle\widehat{n}_b\rangle + \frac{1}{2} > \frac{1}{2}$. Hence for SSR compliant separable states the correlation coefficient is zero. A *quadrature correlation* test for entanglement based on locally SSR compliant sub-system states is then

$$C_{ab}^{\theta\phi} \neq 0 \tag{197}$$

However, it is not difficult to show that for states that are globally SSR compliant

$$\left\langle\widehat{X}_a^\theta\widehat{X}_b^\phi\right\rangle = \left\langle\widehat{S}_x\right\rangle\cos(\theta-\phi) + \left\langle\widehat{S}_y\right\rangle\sin(\theta-\phi) \tag{198}$$

so that the entanglement test based on locally SSR compliant sub-system states is equivalent to finding one of $\left\langle\widehat{S}_x\right\rangle$ or $\left\langle\widehat{S}_y\right\rangle$ to be non-zero. This is the *same* as the previous *Bloch vector* test in Eq.(68) or the weak correlation test in Eq.(166).

6.3 Two Mode Quadrature Squeezing

From Eq. (194) we can define *two mode quadrature operators* as

$$\begin{aligned}
\widehat{X}_\theta(+) &= \frac{1}{\sqrt{2}}(\widehat{X}_a^\theta + \widehat{X}_b^\theta) = \frac{1}{2}(\widehat{a}\exp(-i\theta) + \widehat{b}^\dagger \exp(+i\theta) + \widehat{a}^\dagger \exp(+i\theta) + \widehat{b}\exp(-i\theta)) \\
\widehat{P}_\theta(+) &= \frac{1}{\sqrt{2}}(\widehat{P}_a^\theta + \widehat{P}_b^\theta) = \frac{1}{2i}(\widehat{a}\exp(-i\theta) - \widehat{b}^\dagger \exp(+i\theta) - \widehat{a}^\dagger \exp(+i\theta) + \widehat{b}\exp(-i\theta)) \\
&= \widehat{X}_{\theta+\pi/2}(+)
\end{aligned} \tag{199}$$

where we have $[\ \widehat{X}_\theta(+), \widehat{P}_\theta(+)] = i$. Note that $\widehat{X}_0(+) = (\widehat{x}_A + \widehat{x}_B)/\sqrt{2}$ and $\widehat{P}_0(+) = (\widehat{p}_A + \widehat{p}_B)/\sqrt{2}$ unlike the operators considered in Section 6.1.2. As we have seen there is no entanglement test for systems of identical bosons of the form $\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2\rangle + \langle \Delta(\widehat{p}_A - \widehat{p}_B)^2\rangle < 2 + 2\left\langle \widehat{N}\right\rangle$. The Heisenberg uncertainty principle gives $\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle \left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle \geq 1/4$, so a state is squeezed in $\widehat{X}_\theta(+)$ if $\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle < 1/2$, and similarly for squeezing in $\widehat{P}_\theta(+)$.

We can show that for separable states both $\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle \geq 1/2$ and $\left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle \geq 1/2$, so two mode quadrature squeezing in either $\widehat{X}_\theta(+)$ or $\widehat{P}_\theta(+)$ is a test for two mode entanglement. Firstly, for SSR compliant sub-system states

$$\left\langle \widehat{X}_\theta(+)\right\rangle = \frac{1}{\sqrt{2}}\sum_R P_R(\left\langle \widehat{X}_a^\theta\right\rangle_R + \left\langle \widehat{X}_b^\theta\right\rangle_R) = 0 \tag{200}$$

since $\langle \widehat{a}\rangle_R = \left\langle \widehat{b}\right\rangle_R = 0$. Secondly,

$$\begin{aligned}
&\left\langle \widehat{X}_\theta^2(+)\right\rangle \\
&= \frac{1}{4}\sum_R P_R(\langle \widehat{a}\widehat{a}^\dagger\rangle_R + \langle \widehat{a}^\dagger\widehat{a}\rangle_R + \left\langle \widehat{b}\widehat{b}^\dagger\right\rangle_R + \left\langle \widehat{b}^\dagger\widehat{b}\right\rangle_R) \\
&= \sum_R P_R(\frac{1}{2} + \frac{1}{2}(\langle \widehat{n}_a\rangle_R + \langle \widehat{n}_b\rangle_R) \\
&= \frac{1}{2} + \frac{1}{2}\left\langle \widehat{N}\right\rangle \\
&\geq \frac{1}{2}
\end{aligned} \tag{201}$$

where for local SSR compliant states other terms involving $\langle \widehat{a}^2\rangle_R, \left\langle \widehat{b}^2\right\rangle_R, \left\langle \widehat{a}\widehat{b}\right\rangle_R = \langle \widehat{a}\rangle_R \left\langle \widehat{b}\right\rangle_R, \left\langle \widehat{a}\widehat{b}^\dagger\right\rangle_R = \langle \widehat{a}\rangle_R \left\langle \widehat{b}^\dagger\right\rangle_R$ etc. are all zero. Hence

$$\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle = \left\langle \widehat{X}_\theta^2(+)\right\rangle - \left\langle \widehat{X}_\theta(+)\right\rangle^2 \geq \frac{1}{2} \tag{202}$$

which establishes the result. Since $\widehat{P}_\theta(+) = \widehat{X}_{\theta+\pi/2}(+)$ we also have $\left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle = \frac{1}{2}+\frac{1}{2}\left\langle \widehat{N}\right\rangle \geq \frac{1}{2}$ for a separable state. Hence the *two mode quadrature squeezing* test. If

$$\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle < \frac{1}{2} \qquad or \qquad \left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle < \frac{1}{2} \tag{203}$$

then the state is entangled. Obviously $\widehat{X}_\theta(+)$ and $\widehat{P}_\theta(+)$ cannot both be squeezed for the same state.

We can also define additional two mode quadrature operators as

$$\begin{aligned}
\widehat{X}_\theta(-) &= \frac{1}{\sqrt{2}}(\widehat{X}_a^\theta - \widehat{X}_b^\theta) = \frac{1}{2}(\widehat{a}\exp(-i\theta) - \widehat{b}^\dagger\exp(+i\theta) + \widehat{a}^\dagger\exp(+i\theta) - \widehat{b}\exp(-i\theta)) \\
\widehat{P}_\theta(-) &= \frac{1}{\sqrt{2}}(\widehat{P}_a^\theta - \widehat{P}_b^\theta) = \frac{1}{2i}(\widehat{a}\exp(-i\theta) + \widehat{b}^\dagger\exp(+i\theta) - \widehat{a}^\dagger\exp(+i\theta) - \widehat{b}\exp(-i\theta)) \\
&= \widehat{X}_{\theta+\pi/2}(-)
\end{aligned} \tag{204}$$

where we also have $[\ \widehat{X}_\theta(-), \widehat{P}_\theta(-)] = i$. Again $\left\langle \Delta\widehat{X}_\theta^2(-)\right\rangle\left\langle \Delta\widehat{P}_\theta^2(-)\right\rangle \geq 1/4$, so a state is squeezed in $\widehat{X}_\theta(-)$ if $\left\langle \Delta\widehat{X}_\theta^2(-)\right\rangle < 1/2$, and similarly for squeezing in $\widehat{P}_\theta(-)$.

A similar proof shows that for separable states both $\left\langle \Delta\widehat{X}_\theta^2(-)\right\rangle = \frac{1}{2}+\frac{1}{2}\left\langle \widehat{N}\right\rangle \geq 1/2$ and $\left\langle \Delta\widehat{P}_\theta^2(-)\right\rangle = \frac{1}{2}+\frac{1}{2}\left\langle \widehat{N}\right\rangle \geq 1/2$, so if

$$\left\langle \Delta\widehat{X}_\theta^2(-)\right\rangle < \frac{1}{2} \qquad or \qquad \left\langle \Delta\widehat{P}_\theta^2(-)\right\rangle < \frac{1}{2} \tag{205}$$

then the state is entangled. Hence *any* one of $\widehat{X}_\theta(+), \widehat{P}_\theta(+), \widehat{X}_\theta(-), \widehat{P}_\theta(-)$ being squeezed will demonstate two mode entanglement.

The question then arises - Can two of the four two mode quadrature operators be squeezed? For simplicity we only discuss $\theta = 0$ cases in detail. Obviously pairs such as $\widehat{X}_0(+), \widehat{P}_0(+)$ or $\widehat{X}_0(-), \widehat{P}_0(-)$ cannot. Next, we consider $\widehat{X}_0(+)$ and $\widehat{P}_0(-)$. We note that for all global SSR compliant states $\left\langle \Delta\widehat{X}_0^2(+)\right\rangle + \left\langle \Delta\widehat{P}_0^2(-)\right\rangle = \frac{1}{2}\left(\left\langle \Delta(\widehat{x}_A+\widehat{x}_B)^2\right\rangle + \left\langle \Delta(\widehat{p}_A-\widehat{p}_B)^2\right\rangle\right) = 1+\left\langle \widehat{N}\right\rangle$ using (184), so that if $\widehat{X}_0(+)$ is squeezed $\left\langle \Delta\widehat{X}_0^2(+)\right\rangle < \frac{1}{2}$ then $\left\langle \Delta\widehat{P}_0^2(-)\right\rangle > \frac{1}{2}+\left\langle \widehat{N}\right\rangle$, showing that both $\widehat{X}_0(+)$ and $\widehat{P}_0(-)$ cannot both be squeezed - in spite of the operators commuting. A similar conclusion applies to $\widehat{X}_0(-)$ and $\widehat{P}_0(+)$. For the pair $\widehat{X}_0(+)$ and $\widehat{X}_0(-)$ we have $\left\langle \Delta\widehat{X}_0^2(+)\right\rangle + \left\langle \Delta\widehat{X}_0^2(-)\right\rangle = \frac{1}{2}\left(\left\langle \Delta(\widehat{x}_A+\widehat{x}_B)^2\right\rangle + \left\langle \Delta(\widehat{x}_A-\widehat{x}_B)^2\right\rangle\right) = 1+\left\langle \widehat{N}\right\rangle$ using (182) and (186), so the same situation as for $\widehat{X}_0(+)$ and $\widehat{P}_0(-)$ applies, and thus $\widehat{X}_0(+)$ and $\widehat{X}_0(-)$ cannot both be squeezed. A similar conclusion applies to $\widehat{P}_0(-)$ and $\widehat{P}_0(+)$. In general, only *one* of $\widehat{X}_\theta(+), \widehat{P}_\theta(+), \widehat{X}_\theta(-), \widehat{P}_\theta(-)$ can be squeezed.

Further questions are: What quantities need to be measured in order to test whether two mode quadrature squeezing occurs and how useful would it be to detect entanglement? It is straight-forward to show from (199) and (204) that for states that are global SSR compliant

$$\begin{aligned}
\left\langle \widehat{X}_\theta(+) \right\rangle &= 0 \qquad \left\langle \widehat{X}_\theta(-) \right\rangle = 0 \qquad (206) \\
\left\langle \Delta\widehat{X}_\theta^2(+) \right\rangle &= \left\langle \widehat{X}_\theta^2(+) \right\rangle = \frac{1}{2}\left(\left\langle \widehat{N} \right\rangle + 2\left\langle \widehat{S}_x \right\rangle \right) \\
\left\langle \Delta\widehat{X}_\theta^2(-) \right\rangle &= \left\langle \widehat{X}_\theta^2(-) \right\rangle = \frac{1}{2}\left(\left\langle \widehat{N} \right\rangle - 2\left\langle \widehat{S}_x \right\rangle \right) \qquad (207)
\end{aligned}$$

since terms such as $\langle \widehat{a}^2 \rangle$, $\left\langle \widehat{a}\widehat{b} \right\rangle$ etc are all zero for SSR compliant states. As explained in Section 12.2, both $\left\langle \widehat{N} \right\rangle + 2\left\langle \widehat{S}_x \right\rangle$ and $\left\langle \widehat{N} \right\rangle - 2\left\langle \widehat{S}_x \right\rangle$ are always non-negative, but the entanglement test would require

$$\begin{aligned}
\left\langle \widehat{N} \right\rangle + 2\left\langle \widehat{S}_x \right\rangle &< 1 \qquad \textit{for squeezing in } \widehat{X}_\theta(+) \\
\left\langle \widehat{N} \right\rangle - 2\left\langle \widehat{S}_x \right\rangle &< 1 \qquad \textit{for squeezing in } \widehat{X}_\theta(-) \qquad (208)
\end{aligned}$$

This shows that the two mode quadrature squeezing test involves measuring $\left\langle \widehat{N} \right\rangle$ and $\left\langle \widehat{S}_x \right\rangle$, so that once again measurements of boson number and the mean value of a spin operator are involved. Similar conclusions apply for $\widehat{P}_\theta(+)$ and $\widehat{P}_\theta(-)$. However, since the test requires $\left\langle \widehat{S}_x \right\rangle$ to be non-zero it would simpler to use the *Bloch vector* test (see (68)) which merely requires showing that one of $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ to be non-zero.

In most cases the inequalities in (208) will not be satisfied, since both $\left\langle \widehat{N} \right\rangle$ and $\left\langle \widehat{S}_x \right\rangle$ are $O(N)$. However, for the binomial state in (77) with $\theta = 3\pi/4$ and $\chi = 0$ we have for $|\Phi\rangle = \left(\frac{-\widehat{a}^\dagger + \widehat{b}^\dagger}{\sqrt{2}} \right)^N |0\rangle / \sqrt{N!}$ the results $\left\langle \widehat{N} \right\rangle = N$ and $\left\langle \widehat{S}_x \right\rangle = -N/2$ (see (163) in Ref. [6]). Hence spin squeezing in $\widehat{X}_\theta(+)$ occurs, confirming that this particular binomial state is entangled. Note that the test does not confirm entanglement for almost all other binomial states (those where $\left\langle \widehat{S}_x \right\rangle$ is different from $\pm N/2$), though these are actually entangled.

7 Interferometry in Bosonic Systems

In this section we discuss how interferometers in two mode bosonic systems operate. This topic has of course been discussed many times before, but for completeness we present it here. Our approach is essentially the same as in earlier papers, for example that of Yurke et al [36]. Before discussing interferometry in two mode bosonic systems, we first need to set out the general Hamiltonian for the two mode systems that could be of interest. The two modes may be associated with two distinct single boson *spatial* states, such as in a double well potential in which case the coupling between the two modes is associated with *quantum tunneling*. Or they may be associated with two different atomic internal *hyperfine* states in a single well, which may be coupled via *classical fields* in the form of very short pulses, for which the time dependent amplitude is$\mathcal{A}(t)$, the centre frequency is ω_0 and the *phase* is ϕ. Since this coupling process is much easier to control than quantum tunneling, we will mainly focus on the case of two modes associated with different hyperfine states, though the approach might also be applied to the case of two spatial modes. The free atoms occupying the two modes are associated with energies $\hbar\omega_a$, $\hbar\omega_b$, the *transition frequency* $\omega_{ba} = \omega_b - \omega_a$ being close to *resonance* with ω_0. It is envisaged that a large *number* N of *bosonic atoms* occupy the two modes. The bosonic atoms may also interact with each other via *spin conserving*, *zero range* interatomic potentials. We will show that measurements on the mean and variance for the *population difference* determine the *mean values* and *covariance matrix* for the spin operators involved in *entanglement* tests.

For interferometry involving *multi-mode* systems, a straightforward generalisation of the two mode case is possible, based on the reasonable assumption the interferometer process couples the modes in a pairwise manner. This is based on the operation of *selection rules*, as will be explained below.

However, although in the present section we show that two mode interferometers can be used to measure the *mean values* and *covariance matrix* for the *spin* operators involved in *entanglement* tests for systems of *massive* bosons, the issue of how to measure *mean values* and *variances* for the ***quadrature*** operators involved in other entanglement tests for massive bosons is still to be established. Such a measurement is not possible with the beam splitter interferometer discussed in this paper. An interferometer involving some sort of *phase reference* would seem to be needed. Proposals based on *homodyne* measurements have been made by Olsen et al [37], [38], but these are based on hypothetical reference systems with large numbers of bosons in *Glauber coherent* states, and such states are prohibited via the global particle number SSR.

7.1 Simple Two Mode Interferometer

A simple description of the two mode system is provided by the *Josephson model*, where the overall Hamiltonian is of the form [6]

$$\widehat{H}_{Joseph} = \widehat{H}_0 + \widehat{V} + \widehat{V}_{\mathrm{col}} \tag{209}$$

with

$$
\begin{aligned}
\widehat{H}_0 &= \hbar\omega_a\,\widehat{a}^\dagger\widehat{a} + \hbar\omega_b\,\widehat{b}^\dagger\widehat{b} \\
\widehat{V} &= \mathcal{A}(t)\exp(-i\omega_0 t)\,\exp(i\phi)\,\widehat{b}^\dagger\widehat{a} + \mathcal{A}(t)\exp(+i\omega_0 t)\,\exp(-i\phi)\,\widehat{a}^\dagger\widehat{b} \\
\widehat{V}_{\mathrm{col}} &= \chi(\widehat{b}^\dagger\widehat{b} - \widehat{a}^\dagger\widehat{a})^2
\end{aligned}
\tag{210}
$$

where $\widehat{H}_0$ is the free boson Hamiltonian, $\widehat{V}$ gives the interaction with the classical field and $\widehat{V}_{\mathrm{col}}$ is the collisional interaction term. For the case of quantum tunneling between two distinct *spatial* modes, the interaction term $\widehat{V}$ can also be described in the Josephson model (see [6] for details), in which case the factors multiplying $\widehat{b}^\dagger\widehat{a}$ or $\widehat{a}^\dagger\widehat{b}$ involve the *trapping potential* and the two spatial mode functions. A time dependent amplitude and phase might be obtained via adding a suitable time dependent field to the trapping potential - this would be experimentally difficult. The Hamiltonian can also be written in terms of spin operators as

$$
\begin{aligned}
\widehat{H}_0 &= 1/2(\hbar\omega_a + \hbar\omega_b)\widehat{N} - \hbar\omega_{ab}\widehat{S}_z \\
\widehat{V} &= \mathcal{A}(t)\exp(-i\omega_0 t)\,\exp(i\phi)\,(\widehat{S}_x + i\widehat{S}_y) + H.C \\
\widehat{V}_{\mathrm{col}} &= 4\chi\widehat{S}_z^2
\end{aligned}
\tag{211}
$$

The coupling effect in a *simple two mode interferometer* can be described via the classical interaction term $\widehat{V}$, where now the amplitude $\mathcal{A}(t)$ is only non-zero over a short time interval. The pulsed classical field is associated with an *area variable* s, defined by

$$
s = \int_{t_0}^{t} dt\,_1\mathcal{A}(t_1)/\hbar \tag{212}
$$

the integral eventually being over the pulse's duration. The *interferometer frequency* ω_0 is assumed for simplicity to be in *resonance* with the *transition frequency* $\omega_{ba} = \omega_b - \omega_a$. The classical field is also associated with a phase ϕ, so the simple two mode interferometer is described by two *interferometric variables* $2s = \theta$ giving the pulse area and ϕ specifying the phase. Changing these variables leads to a range of differing applications of the interferometer. When acting as a *beam splitter* (BS) a $2s = \pi/2$ pulse is involved and ϕ is variable, but for a *phase changer* a $2s = \pi$ pulse is involved (ϕ is arbitrary). For *state tomography* in the yz plane we choose $2s = \theta$ (variable) and $\phi = 0$ or π. The beam splitter enables state tomography in the xy plane to be carried out. Generally speaking the effect of *collisions* can be *ignored* during the short classical pulse and we will do so here.

7.2 General Two Mode Interferometers

More complex two mode bosonic interferometers applied to a specific input quantum state will involve specific *arrangements* of simple two mode interferometers such as beam splitters, phase changers and free evolution intervals, followed

by final measurement of the mean population diference between the modes and its variance. *Ramsey interferometry* involves *two* beam splitters separated by a controllable free evolution *time interval* T. During such an interval in which free evolution occurs, the interaction of the classical beam splitter field with the two mode system can be ignored, but the effect of collisions and coupling to external systems may be important if collision parameters are to be measured using the interferometer. The overall behaviour of such multi-element interferometers will also depend on the initial two mode quantum *input* state that acts as the *input state* for the interferometer, as well as important variables such as the phase ϕ, the centre frequency ω_0, the area variable s for the classical pulses used, and also the the free evolution intervals (if any). The behaviour also will depend on the characteristic parameters such as the transition frequency ω_{ab}, collision parameter χ and total boson number N for the two mode system used in the interferometry. The variables that describe the interaction with other systems whose properties are to be measured using the interferometer must also affect its behaviour if the interferometer is to be useful. Finally, a choice must be made for the interferometer physical quantity whose *mean value* and *quantum fluctuation* is to be measured - referred to as the *measurable.* The outcome of such measurements can be studied as a function of one or more of the variables on which the interferometer behaviour depends - referred to as the *interferometric variable.* There are obviously a wide range of possible two mode *interferometers types* that could be studied, depending on the application envisaged. Interferometers also have a wide range of *uses*, including determining the properties of the input two mode state - such as squeezing or entanglement. For a suitable known input state they can be used to measure interferometric variables - such as the classical phase ϕ of the pulsed field associated with a beam splitter or a parameter associated with an external system coupled to the interferometer. On the other hand, in a Ramsey interferometer the interferometric could be the collision parameter χ, obtainable if the free evolution period T is known. No attempt to be comprehensive will be made here.

The *Ramsey interferometer* is illustrated in Figure 7.

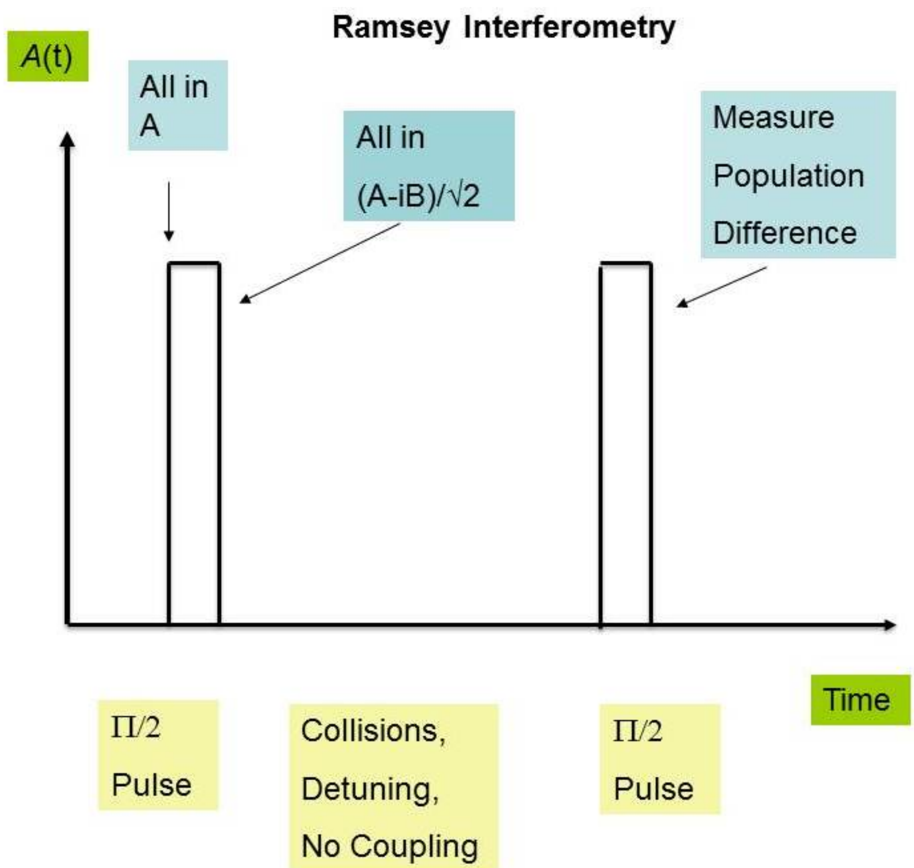

Figure 7. Ramsey Interferometry. Two $\pi/2$ beam splitters separated by a free evolution region.

For the purpose of considering entanglement tests a *simple two mode interferometer* operating under conditions of exact *resonance* $\omega_0 = \omega_{ab}$ will be treated, and its behaviour for N large when the phase ϕ is changed and for different choices of the input state $\widehat{\rho}$ will be examined. Measurements appropriate to detecting entanglement via *spin squeezing* and *correlation* will be discussed. The measurable chosen will initially be half the population difference $(\widehat{b}^{\dagger}\widehat{b} - \widehat{a}^{\dagger}\widehat{a})/2$ - which equals $\widehat{S}_z$ - generally measured after the two mode system has interacted with the simple interferometer, but also without this interaction. The phase ϕ will act as the interferometric variable, as will the pulse area $2s = \theta$. As we will see, different choices of input state ranging from *separable* to *entangled* states lead to markedly different behaviours. In particular, the behaviour of *relative phase eigenstates* as input states will be examined. Later we will also consider measurements involving the square of $\widehat{S}_z$.

7.3 Measurements in Simple Two Mode Interferometer

As discussed in the previous paragraph, the initial choice of *measurable* is

$$\widehat{M} = \frac{1}{2}(\widehat{b}^{\dagger}\widehat{b} - \widehat{a}^{\dagger}\widehat{a}) = \widehat{S}_z \tag{213}$$

and we will determine its mean and variance for the state $\widehat{\rho}^{\#}$ given by

$$\widehat{\rho}^{\#} = \widehat{U}\,\widehat{\rho}\,\widehat{U}^{-1} \tag{214}$$

where the *output state* $\widehat{\rho}^{\#}$ has evolved from the initial *input state* $\widehat{\rho}$ due to the effect of the *simple two mode interferometer*. $\widehat{U}$ is the unitary *evolution operator* describing evolution during the time the short classical pulse is applied. Collision terms and interactions with other systems will be ignored during the short time interval involved.

We note that for an N boson state the eigenvalues of $\widehat{M}$ range from $-N/2$ to $+N/2$ in integer steps. For more general states the possible values for $\widehat{M}$ are any integer or half integer. When $\widehat{M}$ is measured the result will be one of these eigenvalues, but the average of repeated measurements will be $\left\langle \widehat{M} \right\rangle$ which must also lie in the range $-N/2$ to $+N/2$. The variance of the results for the repeated measurements of $\widehat{M}$ is also experimentally measureable and will not exceed $(N/2)^2$, and apart from $NOON$ states will be much less than this. The experimentally determinable results for both $\left\langle \widehat{M} \right\rangle$ and $\left\langle \Delta\widehat{M}^2 \right\rangle$ will depend on $\widehat{\rho}$ and on the interferometer variables such as the phase ϕ and the pulse area $2s = \theta$.

The Hamiltonian governing the evolution in the simple two mode interferometer will be $\widehat{H}_0 + \widehat{V}$. For the *output state* the mean value and variance are

$$\begin{aligned} \left\langle \widehat{M} \right\rangle &= Tr(\widehat{M}\,\widehat{\rho}^{\#}) \\ \left\langle \Delta\widehat{M}^2 \right\rangle &= Tr(\left\{ \widehat{M} - \left\langle \widehat{M} \right\rangle \right\}^2 \widehat{\rho}^{\#}) \end{aligned} \tag{215}$$

These will be evaluated at the end of the pulse. If the input state is measured *directly* without applying the interferometer, then the mean value and variance are as in the last equations but with $\widehat{\rho}^{\#}$ replaced by $\widehat{\rho}$.

The derivation of the results is set out in Appendix 17 and are given by the same form as (215), but with $\widehat{\rho}^{\#}$ replaced by $\widehat{\rho}$ and with $\widehat{M}$ replaced by the interaction picture Heisenberg operator $\widehat{M}_H(2s,\phi)$ at the end of the pulse, which is given by

$$\begin{aligned}\widehat{M}_H(2s,\phi) &= \frac{1}{2}(\widehat{b}_H^{\dagger}(s,\phi)\widehat{b}_H(s,\phi)-\widehat{a}_H^{\dagger}(s,\phi)\widehat{a}_H(s,\phi)) \\ &= \sin 2s\,(\sin\phi\,\widehat{S}_x+\cos\phi\,\widehat{S}_y)+\cos 2s\,\widehat{S}_z \qquad (216)\end{aligned}$$

with

$$\widehat{b}_H(s,\phi)=\cos s\,\widehat{b}-i\exp(i\phi)\,\sin s\,\widehat{a} \qquad \widehat{a}_H(s,\phi)=-i\exp(-i\phi)\,\sin s\,\widehat{b}+\,\cos s\,\widehat{a} \qquad (217)$$

The versatility of the measurement follows from the range of possible choices of the pulse area $2s=\theta$ and the phase ϕ. These results are valid for both *bosonic* and *fermionic* modes.

We then find that the *general result* for the *mean* value is

$$\left\langle \widehat{M}\right\rangle = \sin\theta\,\sin\phi\,\left\langle \widehat{S}_x\right\rangle_{\rho}+\sin\theta\,\cos\phi\,\left\langle \widehat{S}_y\right\rangle_{\rho}+\cos\theta\,\left\langle \widehat{S}_z\right\rangle_{\rho} \qquad (218)$$

and for the *variance* is

$$\begin{aligned}&\left\langle \Delta\widehat{M}^2\right\rangle \\ =\;& \frac{(1-\cos 2\theta)}{2}\frac{(1-\cos 2\phi)}{2}C(\widehat{S}_x,\widehat{S}_x)+\frac{(1-\cos 2\theta)}{2}\frac{(1+\cos 2\phi)}{2}C(\widehat{S}_y,\widehat{S}_y) \\ &+\frac{(1+\cos 2\theta)}{2}C(\widehat{S}_z,\widehat{S}_z) \\ &+\frac{(1-\cos 2\theta)}{2}\sin 2\phi\,C(\widehat{S}_x,\widehat{S}_y)+\sin 2\theta\,\cos\phi\,C(\widehat{S}_y,\widehat{S}_z)+\sin 2\theta\,\sin\phi\,C(\widehat{S}_z,\widehat{S}_x)\end{aligned} \qquad (219)$$

where the *mean value* of the spin operators are $\left\langle \widehat{S}_\alpha\right\rangle_{\rho}=Tr(\widehat{S}_\alpha\widehat{\rho})$ and the *covariance matrix* elements are given by $C(\widehat{S}_\alpha,\widehat{S}_\beta)=1/2\left\langle(\widehat{S}_\alpha\widehat{S}_\beta+\widehat{S}_\beta\widehat{S}_\alpha)\right\rangle_{\rho}-\left\langle \widehat{S}_\alpha\right\rangle_{\rho}\left\langle \widehat{S}_\beta\right\rangle_{\rho}$. The diagonal elements $C(\widehat{S}_\alpha,\widehat{S}_\alpha)=\left\langle \widehat{S}_\alpha{}^2\right\rangle_{\rho}-\left\langle \widehat{S}_\alpha\right\rangle_{\rho}^2=\left\langle \Delta\widehat{S}_\alpha^2\right\rangle$ is the variance. By making appropriate choises of the interferometer variables θ (half the the pulse area) and ϕ (the phase) the mean values of all the spin operators and all elements of the covariance matrrix can be measured. *Tomography* for the spin operators in any selected plane can be carried out.

7.3.1 Tomography in xy Plane - Beam Splitter

In the *beam splitter case* (for state tomography in the xy plane) we choose $2s = \pi/2$ and ϕ (variable) giving

$$\widehat{M}_H(\frac{\pi}{2},\phi) = \sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y \tag{220}$$

and we find that for the output state of the *BS interferometer* the mean value and variance of $\widehat{M}$ are given by

$$\left\langle \widehat{M} \right\rangle = \sin\phi \left\langle \widehat{S}_x \right\rangle_\rho + \cos\phi \left\langle \widehat{S}_y \right\rangle_\rho \tag{221}$$

$$\left\langle \Delta\widehat{M}^2 \right\rangle = \frac{1}{2}(1-\cos 2\phi)\,C(\widehat{S}_x,\widehat{S}_x) + \frac{1}{2}(1+\cos 2\phi)\,C(\widehat{S}_y,\widehat{S}_y) + \sin 2\phi\ C(\widehat{S}_x,\widehat{S}_y) \tag{222}$$

showing the mean value for the measureable $\widehat{M}$ depends sinusoidally on the phase ϕ and the *mean values* of the spin operators $\widehat{S}_x, \widehat{S}_y$. The variance for the measurable depends sinusoidally on 2ϕ and on the *covariance matrix* of the same spin operators. Both the means and covariances of the spin operators $\widehat{S}_x, \widehat{S}_y$ now depend on the input state $\widehat{\rho}$ for the interferometer rather than the output state $\widehat{\rho}^{\#}$.

7.3.2 Tomography in yz Plane

For state tomography in the yz plane we obtain the means and covariances of the spin operators $\widehat{S}_y, \widehat{S}_z$. To do this we choose $2s = \theta$ (variable) and $\phi = 0$ so that

$$\widehat{M}_H(\theta,0) = \sin\theta\,\widehat{S}_y + \cos\theta\,\widehat{S}_z \tag{223}$$

and find that for the output state the mean value and variance of $\widehat{M}$ are given by

$$\left\langle \widehat{M} \right\rangle = \sin\theta \left\langle \widehat{S}_y \right\rangle_\rho + \cos\theta \left\langle \widehat{S}_z \right\rangle_\rho \tag{224}$$

$$\left\langle \Delta\widehat{M}^2 \right\rangle = \frac{1}{2}(1-\cos 2\theta)\,C(\widehat{S}_y,\widehat{S}_y) + \frac{1}{2}(1+\cos 2\theta)\,C(\widehat{S}_z,\widehat{S}_z) + \sin 2\theta\ C(\widehat{S}_y,\widehat{S}_z) \tag{225}$$

A *single* measurement does not of course determine the mean value $\left\langle \widehat{M} \right\rangle$. An average over a large number of *independent repetitions* of the measurement is needed to accurately determine $\left\langle \widehat{M} \right\rangle$ which can then be compared to theoretical predictions. This is a well-known practical issue for the experimenter that we need not dwell on here. A brief account of the issues involved is included in Appendix 18.

7.3.3 Phase Changer

In this case we choose $2s = \theta = \pi$ and ϕ (arbitrary) giving

$$\widehat{M}_H(\pi, \phi) = -\widehat{S}_z \tag{226}$$

and for the output state the mean value and variance of $\widehat{M}$ are given by

$$\begin{aligned} \left\langle \widehat{M} \right\rangle &= -\left\langle \widehat{S}_z \right\rangle_\rho & (227) \\ \left\langle \Delta \widehat{M}^2 \right\rangle &= \left\langle \Delta \widehat{S}_z^2 \right\rangle & (228) \end{aligned}$$

so the phase changer measures the negative of the population difference. Effectively the phase changer interchanges the modes $\widehat{a} \to \widehat{b}$ and $\widehat{b} \to \widehat{a}$ and this is its role rather than being directly involved in a measurement. Phase changers are often included in complex interferometers at the midpoint of free evolution regions to cancel out unwanted causes of phase change.

7.3.4 Other Measurements in Simple Two Mode Interferometer

Another useful choice of measureable is the *square* of the population difference

$$\widehat{M}_2 = \left(\frac{1}{2} (\widehat{b}^\dagger \widehat{b} - \widehat{a}^\dagger \widehat{a}) \right)^2 = \widehat{S}_z^2 \tag{229}$$

For the beam splitter case with $2s = \pi/2$ and ϕ (variable), we can easily show (see Appendix 17) that the mean value of $\widehat{M}_2$ for the output state is given by

$$\left\langle \widehat{M}_2 \right\rangle = \sin^2 \phi \left\langle (\widehat{S}_x)^2 \right\rangle + \cos^2 \phi \left\langle (\widehat{S}_y)^2 \right\rangle + \sin \phi \cos \phi \left\langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \right\rangle \tag{230}$$

showing that the mean for the new observable $\widehat{M}_2$ is a sinusoidal function of the BS interferometer variable ϕ with coefficients that depend on the means of $\widehat{S}_x^2$, $\widehat{S}_y^2$ and $\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x$.

Choosing special cases for the interferometer variable yields the following useful results

$$\begin{aligned} \left\langle \widehat{M}_2 \right\rangle_{\phi=0} &= \left\langle (\widehat{S}_y)^2 \right\rangle_\rho \qquad \left\langle \widehat{M}_2 \right\rangle_{\phi=\pi/2} = \left\langle (\widehat{S}_x)^2 \right\rangle_\rho \\ \left\langle \widehat{M}_2 \right\rangle_{\phi=\pi/4} - \left\langle \widehat{M}_2 \right\rangle_{\phi=-\pi/4} &= \left\langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \right\rangle_\rho & (231) \end{aligned}$$

Hence all three quantities $\left\langle (\widehat{S}_x)^2 \right\rangle$, $\left\langle (\widehat{S}_y)^2 \right\rangle$ and $\left\langle (\widehat{S}_x \widehat{S}_y + \widehat{S}_y \widehat{S}_x) \right\rangle$ can be measured. We note that just measuring $\left\langle \widehat{M}_2 \right\rangle$ does not add to the results obtained by measuring the mean and variance of the original measureable $\widehat{M}$, since $\left\langle \widehat{M}_2 \right\rangle = \left\langle \Delta \widehat{M}^2 \right\rangle + \left\langle \widehat{M} \right\rangle^2$. The variance $\left\langle \Delta \widehat{M}_2^2 \right\rangle$ does of course depend on higher moments, for example with $\phi = 0$ $\left\langle \Delta \widehat{M}_2^2 \right\rangle = \left\langle \Delta \left(\widehat{S}_y^2 \right)^2 \right\rangle$ and $\phi = \pi/2$ $\left\langle \Delta \widehat{M}_2^2 \right\rangle = \left\langle \Delta \left(\widehat{S}_x^2 \right)^2 \right\rangle$, so these also could be measured.

7.4 Multi-Mode Interferometers

For the multi-mode case we consider two sets of modes $\widehat{a}_i$ and $\widehat{b}_i$ as descibed in SubSection 2.2. These may be different modes associated with two different hyperfine states or they may be modes associated with two separated potential wells. The Hamiltonian analogous to that in (210) for the two mode case is

$$\begin{aligned}
\widehat{H}_0 &= \sum_i \hbar(\omega_a + \omega_i)\, \widehat{a}_i^\dagger \widehat{a}_i + \sum_i \hbar(\omega_b + \omega_i)\, \widehat{b}_i^\dagger \widehat{b}_i \\
\widehat{V} &= \mathcal{A}(t) \exp(-i\omega_0 t)\, \exp(i\phi) \sum_i \widehat{b}_i^\dagger \widehat{a}_i + \mathcal{A}(t) \exp(+i\omega_0 t)\, \exp(-i\phi) \sum_i \widehat{a}_i^\dagger \widehat{b}_i
\end{aligned} \tag{232}$$

where the collision terms are ignored since we are only considering the effect of the short interferometer coupling pulse. Here we have assumed that the energy for the mode $\widehat{a}_i$ is $\hbar(\omega_a + \omega_i)$, which is the sum of a basic energy for all a modes - $\hbar\omega_a$, and an energy term $\hbar\omega_i$ that distinguishes differing modes $\widehat{a}_i$ (and similarly for the mode $\widehat{b}_i$). In addition, we assume selection rules lead to pairwise coupling $\widehat{a}_i \leftrightarrow \widehat{b}_i$. In the case where coupling is due to pulsed external fields (microwave and RF) we can assume that the momenta ($\sim \sqrt{m\hbar\omega_{trap}}$) associated with trapped modes $\widehat{a}_i$ and $\widehat{b}_i$ are the same, since the momenta associated with the low frequency photons ($\sim \hbar\omega_{RF}/c$) involved can be ignored. The spin operators for the multi-mode system are set out in (6) in terms of the mode operators.

As in SubSection 7.1 the *interferometer frequency* ω_0 is assumed for simplicity to be in *resonance* with the *transition frequency* $\omega_{ba} = \omega_b - \omega_a$. Following the treatment in SubSection 7.3, the choice of *measurable* is the half the total population difference between the two sets of modes

$$\widehat{M} = \frac{1}{2} \sum_i (\widehat{b}_i^\dagger \widehat{b}_i - \widehat{a}_i{}^\dagger \widehat{a}_i) = \widehat{S}_z \tag{233}$$

and we will determine its mean and variance for the state $\widehat{\rho}^{\#}$ given by

$$\widehat{\rho}^{\#} = \widehat{U}\, \widehat{\rho}\, \widehat{U}^{-1} \tag{234}$$

where the *output state* $\widehat{\rho}^{\#}$ has evolved from the initial *input state* $\widehat{\rho}$ due to the effect of the *multi- mode interferometer*. $\widehat{U}$ is the unitary *evolution operator* describing evolution during the time the short classical pulse is applied. Collision terms and interactions with other systems will be ignored during the short time interval involved.

As in the two mode interferometer case, the results for the mean and variance of $\widehat{M}$ depend on the *pulse area* $2s = \theta$ and the *phase* ϕ.of the interferometer coupling pulse. They have the same dependence on the *mean values* and *covariance matrix* for the *multi-mode spin operators* $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ for the input state $\widehat{\rho}$ as in the two mode interferometer. Thus we then find that the *general result* for the *mean* value is

$$\left\langle \widehat{M} \right\rangle = \sin\theta \, \sin\phi \, \left\langle \widehat{S}_x \right\rangle_\rho + \sin\theta \, \cos\phi \, \left\langle \widehat{S}_y \right\rangle_\rho + \cos\theta \, \left\langle \widehat{S}_z \right\rangle_\rho \tag{235}$$

and for the *variance* is

$$
\begin{aligned}
&\left\langle \Delta\widehat{M}^2 \right\rangle \\
= \quad & \frac{(1-\cos 2\theta)}{2}\frac{(1-\cos 2\phi)}{2}C(\widehat{S}_x,\widehat{S}_x)+\frac{(1-\cos 2\theta)}{2}\frac{(1+\cos 2\phi)}{2}C(\widehat{S}_y,\widehat{S}_y) \\
&+\frac{(1+\cos 2\theta)}{2}C(\widehat{S}_z,\widehat{S}_z) \\
&+\frac{(1-\cos 2\theta)}{2}\sin 2\phi\, C(\widehat{S}_x,\widehat{S}_y)+\sin 2\theta\,\cos\phi\, C(\widehat{S}_y,\widehat{S}_z)+\sin 2\theta\,\sin\phi\, C(\widehat{S}_z,\widehat{S}_x)
\end{aligned}
\tag{236}
$$

Details of the derivation are set out in Appendix 17.

7.5 Application to Spin Squeezing Tests for Entanglement

Unless stated otherwise, we now focus on spin squeezing tests for *SSR compliant entanglement* based on the beam splitter measurements (the simple two mode interferometer with $2s = \theta = \pi/2$). By choosing the phase $\phi = 0$ we see that $\left\langle \widehat{M} \right\rangle = \left\langle \widehat{S}_y \right\rangle_\rho$ and $\left\langle \Delta\widehat{M}^2 \right\rangle = C(\widehat{S}_y,\widehat{S}_y) = \left\langle \{\widehat{S}_y - \left\langle \widehat{S}_y \right\rangle_\rho\}^2 \right\rangle_\rho$ giving the mean and variance for the spin operator $\widehat{S}_y$. By choosing the phase $\phi = \pi/2$ we see that $\left\langle \widehat{M} \right\rangle = \left\langle \widehat{S}_x \right\rangle_\rho$ and $\left\langle \Delta\widehat{M}^2 \right\rangle = C(\widehat{S}_x,\widehat{S}_x) = \left\langle \{\widehat{S}_x - \left\langle \widehat{S}_x \right\rangle_\rho\}^2 \right\rangle_\rho$ giving the mean and variance for the spin operator $\widehat{S}_x$. If the measurement of $\left\langle \widehat{M} \right\rangle$ were carried out without the beam splitter being present then $\left\langle \widehat{M} \right\rangle = \left\langle \widehat{S}_z \right\rangle_\rho$. Combining all these results then enables us to see whether or not $\widehat{S}_x$ is squeezed with respect to $\widehat{S}_y$ or vice versa. This illustrates the *use* of the interferometer in seeing if a state $\widehat{\rho}$ is *squeezed*. Squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_y$ (or $\widehat{S}_x$) or vice versa also demonstrates entanglement and again the simple two mode interferometer with appropriate choices of θ and ϕ can be used to measure the means and variances of the relevant spin operators.

As the presence of spin squeezing shows that the state must be entangled [1] the use of the interferometer for squeezing tests is important. Note that we still need to consider whether fluctuations due to a finite measurement sample could mask the test. However, as spin squeezing has been demonstrated in two mode systems of bosonic atoms this approach to demonstrating entanglement is clearly useful.

7.5.1 Spin Squeezing in $\widehat{S}_x$, $\widehat{S}_y$

To demonstrate spin squeezing in $\widehat{S}_x$ with respect to $\widehat{S}_y$ we need to measure the variances $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho$ and $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho$ and the mean $\left\langle \widehat{S}_z \right\rangle_\rho$ and show that

$$\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \qquad \left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho > \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \tag{237}$$

As we have seen, the variances in $\widehat{S}_y$, $\widehat{S}_x$ are obtained by measuring the *fluctuation* in the measureable $\widehat{M}$ *after* applying the interferometer to the state $\widehat{\rho}$, with the interferometer phase set to $\phi = 0$ or $\phi = \pi/2$ for the two cases respectively. The mean $\left\langle \widehat{S}_z \right\rangle_\rho$ is obtained by a direct measurement of the measureable $\widehat{M}$ *without* applying the interferometer to the state $\widehat{\rho}$.

7.5.2 Spin Squeezing in xy Plane

As shown in Section 3 (see [1]) squeezing in $\widehat{S}_x$ compared to $\widehat{S}_y$ or vice versa is sufficient to show that the state is entangled. However, as the interferometer measures the variance for the state $\widehat{\rho}$ in the quantity

$$\widehat{M}_H(\pi/2, \phi) = \sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y = \widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi) \tag{238}$$

corresponding to the x component of the spin vector operator $\underrightarrow{(\widehat{S})}$ after it has been rotated about the z axis through an angle $\frac{3\pi}{2} + \phi$, it is desirable to *extend* the entanglement test to consider the squeezing of $\widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi)$ with respect to the corresponding y component $\widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)$ - and vice versa, where

$$\widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi) = -\cos\phi\,\widehat{S}_x + \sin\phi\,\widehat{S}_y \tag{239}$$

The variance in $\widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)$ can be obtained by changing the interferometer phase to $\phi + \frac{\pi}{2}$. Clearly $[\widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi), \widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)] = i\,\widehat{S}_z$, as before.

The question is - does squeezing in either $\widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi)$ or $\widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)$ demonstrate entanglement of the modes $\widehat{a}$ and $\widehat{b}$? The answer is that it does.

For a *separable* state we have

$$\left\langle \widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi) \right\rangle_\rho = \left\langle \widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi) \right\rangle_\rho = 0 \tag{240}$$

as before, since $\left\langle \widehat{S}_{x,y}^{\#}(\frac{3\pi}{2} + \phi) \right\rangle_\rho$ are just linear combinations of the zero $\left\langle \widehat{S}_{x,y} \right\rangle_\rho$.

Since $[\ \widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi), \widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)] = i\widehat{S}_z$ the Heisenberg uncertainty principle shows that $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi)^2 \right\rangle_\rho \left\langle \Delta\widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)^2 \right\rangle_\rho \geq \frac{1}{4}|\left\langle \widehat{S}_z \right\rangle_\rho|^2$ so spin squeezing in $\widehat{S}_x^{\#}(\frac{3\pi}{2} + \phi)$ with respect to $\widehat{S}_y^{\#}(\frac{3\pi}{2} + \phi)$ or vice versa requires us to show

that

$$\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \qquad or \qquad \left\langle \Delta\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \tag{241}$$

Since we measure $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho$ the spin squeezing test is $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$.

Now for $\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$ we have

$$\begin{aligned} \left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho &= \left\langle \widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho \qquad (242) \\ &= \sin^2\phi \left\langle \widehat{S}_x^2 \right\rangle + \cos^2\phi \left\langle \widehat{S}_y^2 \right\rangle + \sin\phi\,\cos\phi \left\langle \widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x \right\rangle_\rho \end{aligned}$$

and for a separable state we have from SubSection 2.4

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle &= \left\langle \Delta\widehat{S}_x^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \\ \left\langle \widehat{S}_y^2 \right\rangle &= \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \qquad (243) \end{aligned}$$

whilst for the remaining term

$$\begin{aligned} \left\langle \widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x \right\rangle_\rho &= \frac{1}{2i}\left\langle \{(\widehat{b}^\dagger)^2(\widehat{a})^2 - (\widehat{a}^\dagger)^2(\widehat{b})^2\} \right\rangle \\ &= \frac{1}{2i}\sum_R P_R\{\left\langle (\widehat{b}^\dagger)^2 \right\rangle_{\rho_R^B} \left\langle (\widehat{a})^2 \right\rangle_{\rho_R^A} - \left\langle (\widehat{a}^\dagger)^2 \right\rangle_{\rho_R^A} \left\langle (\widehat{b})^2 \right\rangle_{\rho_R^B}\} \\ &= 0 \qquad (244) \end{aligned}$$

using the *local particle number* SSR.

Thus as $\sin^2\phi + \cos^2\phi = 1$ and applying similar considerations to $\left\langle \Delta\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho$

$$\begin{aligned} \left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho &\geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \\ \left\langle \Delta\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho &\geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \qquad (245) \end{aligned}$$

showing that for a separable state there is no squeezing for $\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$ compared to $\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)$ or vice versa. Hence squeezing in either $\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$ or $\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)$ demonstrates entanglement of the modes $\widehat{a}$ and $\widehat{b}$.

7.5.3 Measurement of $\left\langle \widehat{S}_z \right\rangle_\rho$

The question remaining is whether the mean value $\left\langle \widehat{S}_z \right\rangle_\rho$ can be measured accurately enough to apply the test for entanglement. With an infinite number

of repeated measurements this is always possible, since then both the variances $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho$ and the mean $\left\langle \widehat{S}_z \right\rangle_\rho$. would become error free. For a finite number of measurements R the measurement of $\left\langle \widehat{S}_z \right\rangle_\rho$ requires a consideration of the variance in $\widehat{S}_z$. For *general* entangled states general considerations indicate that this mean will be of order N and the variance will be at worst of order N^2, Hence the variance $\left\langle \Delta \left\langle \widehat{S_z} \right\rangle^2 \right\rangle_{sample}$ in determining the mean $\left\langle \widehat{S}_z \right\rangle$ for R repetitions of the measurement would be $\sim N^2/R$, giving a fluctuation of $\sim N/\sqrt{R}$. For this to be small compared to $\sim N$ we merely require $R \gg 1$, which is not unexpected. This result indicates that the application of the spin squeezing test via interferometric measurement of both the variances $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho$ and the mean $\left\langle \widehat{S}_z \right\rangle_\rho$ looks feasible.

7.5.4 Spin Squeezing in $\widehat{S}_z$, $\widehat{S}_y$

To demonstrate spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_y$ we need to measure the variances $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle_\rho$ and $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho$ and the mean $\left\langle \widehat{S}_x \right\rangle_\rho$ and show that

$$\left\langle \Delta\widehat{S}_z{}^2 \right\rangle_\rho < \frac{1}{2} |\left\langle \widehat{S}_x \right\rangle_\rho| \qquad \left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho > \frac{1}{2} |\left\langle \widehat{S}_x \right\rangle_\rho| \tag{246}$$

As we have seen, the variances in $\widehat{S}_z$, $\widehat{S}_y$ are obtained by measuring the *fluctuation* in the measureable $\widehat{M}$ *after* applying the interferometer to the state $\widehat{\rho}$, with the interferometer phase set to $\phi = 0$ and the pulse area $2s = \theta$ made variable. From Eq.(225) we see that choosing $\theta = 0$ gives $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle_\rho$ and choosing $\theta = \frac{\pi}{2}$ gives $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho$. From Eq.(221) the mean $\left\langle \widehat{S}_x \right\rangle_\rho$ is obtained by a measurement of the *mean* in the measureable $\widehat{M}$ *after* applying the interferometer to the state $\widehat{\rho}$, with the interferometer phase set to $\phi = \pi/2$ and the pulse area $2s = \pi/2$.

7.6 Application to Correlation Tests for Entanglement

7.6.1 First Order Correlation Test

Unless stated otherwise, we again focus on correlation tests for *SSR compliant entanglement* For the beam splitter case and by choosing the phase $\phi = 0$ we see that $\left\langle \widehat{M} \right\rangle = \left\langle \widehat{S}_y \right\rangle_\rho$ and by choosing the phase $\phi = \pi/2$ we see that $\left\langle \widehat{M} \right\rangle = \left\langle \widehat{S}_x \right\rangle_\rho$. The simplest form of the correlation test with $n = m = 1$ requires

$$\left\langle \widehat{S}_x \right\rangle_\rho \neq 0 \qquad \left\langle \widehat{S}_y \right\rangle_\rho \neq 0 \tag{247}$$

for establishing that the state is entangled. For the separable state with $\widehat{M}_H = \sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y = \widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$

$$\left\langle \widehat{M} \right\rangle = \left\langle \widehat{M}_H \right\rangle_{\rho} = 0 \tag{248}$$

so that the mean value of the measureable is zero and independent of the beam splitter phase ϕ.for all ϕ. Finding any non-zero value for $\left\langle \widehat{M}_H \right\rangle_{\rho}$ would then show that the state $\widehat{\rho}$ is entangled. More importantly from the general result, $\left\langle \widehat{M}_H \right\rangle_{\rho}$ would show a *sinusoidal dependence* on the interferometer phase ϕ, so the appearance of such a dependence would be an indication that the state was entangled.

The question remaining is whether the mean values $\left\langle \widehat{S}_{x,y} \right\rangle_{\rho}$ can be measured accurately enough to apply the test for entanglement. With an infinite number of repeated measurements this is always possible, since then both the variances $\left\langle \Delta\widehat{S}^2_{x,y} \right\rangle_{\rho}$ and the means $\left\langle \widehat{S}_{x,y} \right\rangle_{\rho}$. would become error free. For a finite number of measurements the measurement of $\left\langle \widehat{S}_{x,y} \right\rangle_{\rho}$ requires a consideration of the variances in $\widehat{S}_{x,y}$ (or $\widehat{M}_H$ to cover both cases). For a general entangled state we can assume that $\left\langle \widehat{M}_H \right\rangle_{\rho} \sim N/2$ and the variance will be at worst of order N^2, Hence the variance $\left\langle \Delta \left\langle \widehat{M}_H \right\rangle^2 \right\rangle_{sample}$ in determining the mean $\left\langle \widehat{S}_{x,y} \right\rangle$ for R repetitions of the measurement would be $\sim N^2/R$, giving a fluctuation of $\sim N/\sqrt{R}$. For this to be small compared to $\sim N$ we merely require $R \gg 1$, which is not unexpected. This result indicates that the application of the simple correlation test via interferometric measurement of $\left\langle \widehat{S}_x \right\rangle_{\rho}$ and $\left\langle \widehat{S}_y \right\rangle_{\rho}$ looks feasible.

7.6.2 Second Order Correlation Test

For the second order form of the correlation test with $n = m = 2$ requires

$$\begin{aligned} \left\langle \Delta\widehat{S}_x{}^2 \right\rangle_{\rho} + \left\langle \widehat{S}_x \right\rangle_{\rho}^2 - \left\langle \Delta\widehat{S}_y{}^2 \right\rangle_{\rho} - \left\langle \widehat{S}_y \right\rangle_{\rho}^2 &\neq 0 \\ (C(\widehat{S}_x, \widehat{S}_y) + \left\langle \widehat{S}_x \right\rangle_{\rho} \left\langle \widehat{S}_y \right\rangle_{\rho}) &\neq 0 \end{aligned} \tag{249}$$

We have already shown using Eqs (221) and (222) that the variances $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_{\rho}$ and $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_{\rho}$ and the means $\left\langle \widehat{S}_y \right\rangle_{\rho}$ and $\left\langle \widehat{S}_x \right\rangle_{\rho}$ can be obtained via the BS interferometer.from the mean $\left\langle \widehat{M} \right\rangle$ and the variance $\left\langle \Delta\widehat{M}^2 \right\rangle$ for the choices of $\phi = 0$ and $\phi = \pi/2$. To obtain the covariance matrix element $C(\widehat{S}_x, \widehat{S}_y)$ we see

that if we choose $\phi = \pi/4$ then $\left\langle \Delta\widehat{M}^2 \right\rangle = \frac{1}{2}\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho + \frac{1}{2}\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho + C(\widehat{S}_x, \widehat{S}_y)$, from which the covariance can be measured. Thus the second order correlation test can be applied.

Alternately, if the measurement quantity for the BS interferometer is the square of the population difference then we see from (231) that the mean value of $\widehat{M}_2$ for certain choices of the BS variable ϕ measures $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho + \left\langle \widehat{S}_x \right\rangle_\rho^2 = \left\langle \widehat{S}_x{}^2 \right\rangle_\rho$, $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho + \left\langle \widehat{S}_y \right\rangle_\rho^2 = \left\langle \widehat{S}_y{}^2 \right\rangle_\rho$ and $(C(\widehat{S}_x, \widehat{S}_y) + \left\langle \widehat{S}_x \right\rangle_\rho \left\langle \widehat{S}_y \right\rangle_\rho) = \left\langle (\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x) \right\rangle_\rho$. The second order correlation test is that if

$$\begin{aligned} \left\langle \widehat{S}_x{}^2 \right\rangle_\rho &\neq \left\langle \widehat{S}_y{}^2 \right\rangle_\rho \\ \left\langle (\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x) \right\rangle_\rho &\neq 0 \end{aligned} \tag{250}$$

then the state is entangled.

7.7 Application to Quadrature Tests for Entanglement

As we saw previously, no useful quadrature test for SSR compliant entanglement in two mode systems of identical bosons of the form $\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle_\rho + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle_\rho < 2 + 2\left\langle \widehat{N} \right\rangle_\rho$ results if the overall system state is globally SSR compliant. However, if the *separable* states are non-compliant then showing that

$$\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle_\rho + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle_\rho < 2 + 2\left\langle \widehat{N} \right\rangle_\rho + 2(\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho + \left\langle \widehat{a}^\dagger\,\widehat{b}^\dagger \right\rangle_\rho) - 2|\left\langle \widehat{a} \right\rangle_\rho + \left\langle \widehat{b}^\dagger \right\rangle_\rho|^2. \tag{251}$$

would demonstrate entanglement. This test requires measuring $\left\langle \widehat{N} \right\rangle_\rho$ together with $\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho$, $\left\langle \widehat{a} \right\rangle_\rho$ and $\left\langle \widehat{b}^\dagger \right\rangle_\rho$. Although the first can be measured using the BS interferometer the quantities $\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho$, $\left\langle \widehat{a} \right\rangle_\rho$ and $\left\langle \widehat{b}^\dagger \right\rangle_\rho$ cannot. Another technique involving a measuring system where there is a well-defined *phase reference* is therefore required if quadrature tests for SSR neglected entanglement are to be undertaken. Furthermore, the *overall* state must still be globally SSR compliant, and hence $\left\langle \widehat{a}\,\widehat{b} \right\rangle_\rho$, $\left\langle \widehat{a} \right\rangle_\rho$ and $\left\langle \widehat{b}^\dagger \right\rangle_\rho$ are all zero, even for entangled states, so the test reduces to $\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle_\rho + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle_\rho < 2 + 2\left\langle \widehat{N} \right\rangle_\rho$. Since for *all* states $\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle_\rho + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle_\rho = 2 + 2\left\langle \widehat{N} \right\rangle_\rho$ this test must fail anyway. We have also seen that finding the *correlation coefficient* - defined in terms of generalised quadrature operators (194) in Eq. (196) - to be non-zero does not lead to a new test for SSR compliant entanglement. The tests involving *two mode quadrature squeezing* look more promising, assuming the relevant quadrature variances can be measured.

8 Experiments on Spin Squeezing

We now examine a number of recent experimental papers involving squeezing and entanglement in BEC with *large* numbers of *identical* bosons. Their notation will be modified to be the same as here. There are really two questions to consider. One is whether squeezing has been created (and of which type). The second is whether or not this demonstrates entanglement of the modes involved. Here we define entanglement for identical bosons as set out in Section **3 of paper I**. Many of these experiments involve Ramsey interferometers and the aim was to demonstrate spin squeezing created via the collisional interaction between the bosons. Obviously, in demonstrating *spin squeezing* they would hope to have created an entangled state, though in most cases an entangled state had already been created via the interaction with the first beam splitter. Although the criterion for entanglement used in most cases was based on an experimental *proposal* [13], [31] which regarded identical particles as *distinguishable* sub-systems, the spin squeezing test based on $\widehat{S}_z$ does turn out to be a.valid test for two mode entanglement, as explained in Section 3. However, it should be noted that all the papers discussed have a different viewpoint regarding what exactly is entangled - generally referring to entanglement of *atoms* or *particles* rather than modes. All the experiments discussed below establish entanglement, though often this was already created in a first $\pi/2$ coupling pulse. Most are based on the spin squeezing test involving $\widehat{S}_z$, that of Gross et al [39] involved *population difference squeezing* rather than spin squeezing (see SubSection 3.5). The other experiment of Gross et al [40] shows (see Fig 2b in [40]) that the mean value of one of the two spin operators $\widehat{S}_x$, $\widehat{S}_y$ is non-zero, as measurement results such as in (224) for the simple two mode interferometer with $2s = \theta$, $\phi = 0$ can determine. This is sufficient to demonstrate two mode entanglement, as (68) shows.

A key result of the present paper (and [1]) is that the conclusion that experiments which have demonstrated spin squeezing in $\widehat{S}_z$ have thereby demonstrated two mode entanglement, no longer has to be justified on the basis of a proof that clearly does not apply to a system of identical bosons.

8.1 Esteve et al. (2008) [41]

• Stated emphasis - generation of spin squeezed states suitable for atom interferometry, demonstration of *particle* entanglement.

• System - Rb^{87} in two hyperfine states.

• BEC of Rb^{87} trapped in optical lattice superposed on harmonic trap.

• Occupation number per site 100 to 1100 atoms - macroscopic.

• Situation where atoms trapped in just two sites treated - two mode entanglement?

• Claimed observed (see Fig 1 in [41]) spin squeezing based on $N\left\langle \Delta\widehat{S}_z^2\right\rangle/(\left\langle \widehat{S}_x^2\right\rangle+\left\langle \widehat{S}_y^2\right\rangle) < 1$ (see (132)).

• Claimed entanglement of identical atoms.

• Spin squeezing test is based on assumption that Bloch vector is on Bloch sphere, a result not established.

• Comment - spin squeezing in $\widehat{S}_z$ (almost) demonstrated (see (132)), so entanglement is established.

8.2 Riedel et al. (2010) [42]

• Stated emphasis - generation of spin squeezed states suitable for atom interferometry, demonstration of *multi-particle* entanglement.

• System - Rb^{87} in two hyperfine states.

• BEC of Rb^{87} trapped in harmonic trap with non-zero magnetic field - Zeeman splitting of levels.

• Number of atoms 1200 - macroscopic.

• Process involves Ramsey interferometry - starts with all atoms in one state, $\pi/2$ pulse (duration $\pi/2\Omega$?) generates coherent spin state $(\widehat{a}^\dagger + \widehat{b}^\dagger)^N |0\rangle$ (entangled), free evolution with collisions (causes squeezing), second pulse with area $2s = \theta$ and phase π (or 0) followed by detection of population difference - associated with operator $\widehat{S}_z$.

• Evolution described using Josephson Hamiltonian $\widehat{H} = \delta\widehat{S}_z + \Omega\widehat{S}_\phi + \chi\widehat{S}_z^2$ where $\widehat{S}_\phi = \cos\phi\,\widehat{S}_x - \sin\phi\,\widehat{S}_y$, Ω is Rabi frequency, ϕ is phase of RF-microwave field, δ is detuning, χ describes collisions. Interaction picture and on resonance ?

• During free evolution including collisions spatial modes for internal states pushed apart so that χ becomes much bigger in order to give larger squeezing.

• Final pulse enables state tomography in the yz plane to be carried out - measures spin squeezing for spin operator $\widehat{S}_\theta = \cos\theta\,\widehat{S}_z - \sin\theta\,\widehat{S}_y$ in this plane (see (236) herein with $\phi = \pi$).

• Claimed observed spin squeezing based on $\left\langle \Delta\widehat{S}_\theta^2 \right\rangle$ being less than standard quantum limit $N/4$. (see Fig 2 in [42]).

• No measurement made to show that $|\left\langle \widehat{S}_x \right\rangle| \approx N/2$ as required to justify spin squeezing test. Spin squeezing test is based on assumption that Bloch vector is on Bloch sphere, a result not established.

• Claim that state of atoms at end of free evolution is four-partite entangled based on spin squeezing test is not substantiated, also an entangled state was already created by first $\pi/2$ pulse.

• Comment - spin squeezing in $\widehat{S}_z$ (almost) demonstrated (see (132)), so entanglement is established. An entangled state was of course already created by first $\pi/2$ pulse, and then modified via the collisional effects.

8.3 Gross et al. (2010) [40]

• Stated emphasis - generation of non-classical spin squeezed states for nonlinear atom inteferometry, demonstration of entanglement between *atoms.*

• System - Rb^{87} in two hyperfine states.

• Six independent BECs of Rb^{87} trapped in six separate wells in a optical lattice.

• Number of atoms 2300 - macroscopic, down to ca 170 in each well.

• Evolution described using Josephson Hamiltonian $\widehat{H} = \Delta\omega_0\widehat{S}_z + \Omega\widehat{S}_\gamma + \chi\widehat{S}_z^2$ where (in the present notation) $\widehat{S}_\gamma = \cos\gamma\,\widehat{S}_x + \sin\gamma\,\widehat{S}_y$, Ω is Rabi frequency, γ is phase of RF-microwave field, $\Delta\omega_0$ is detuning, χ describes collisions. Interaction picture and on resonance ?

• During free evolution plus collision evolution Feshach resonance used so that χ becomes much bigger in order to give larger squeezing.

• One process involves Ramsey interferometry - starts with all atoms in one state, $\pi/2$ pulse (duration $\pi/2\Omega$?) generates coherent spin state $(\widehat{a}^\dagger + \widehat{b}^\dagger)^N |0\rangle$ (entangled) with $\left\langle \widehat{S}_z \right\rangle = 0$, free evolution with collisions (causes squeezing) and with spin echo pulse applied, second $\pi/2$ pulse with phase ϕ followed by detection of population difference - associated with operator $\widehat{S}_z$.

• Population difference measured after last $\pi/2$ pulse shows a sinusoidal dependence on phase ϕ (see Fig 2b in [40]). This shows that $\left\langle \widehat{S}_x \right\rangle$ and $\left\langle \widehat{S}_y \right\rangle$ are non-zero, thereby showing that the state created just prior to last pulse is entangled (see Bloch vector test (68)). This does not of course show that the state is spin squeezed.

• Another process involves generation of coherent spin state $(\widehat{a}^\dagger + \widehat{b}^\dagger)^N |0\rangle$ (entangled) with $\left\langle \widehat{S}_z \right\rangle = 0$, then free evolution with collisions (causes squeezing), followed by coupling pulse to rotate Bloch vector through angle α thereby crossing xy plane. The variance in $\widehat{S}_z$ is then measured as α changes.

• Claimed observed spin squeezing based on $N\left\langle \Delta\widehat{S}_z^2 \right\rangle / (\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2)$ being less than 1 (see Fig 3 in [40]).

• Spin squeezing test is based on assumption that Bloch vector is on Bloch sphere, a result not established since $\left\langle \widehat{S}_x \right\rangle$ and $\left\langle \widehat{S}_y \right\rangle$ not measured.

• Claimed entanglement of ca 170 atoms.

• Comment - spin squeezing in $\widehat{S}_z$ (almost) demonstrated (see (132)), so entanglement is established. An entangled state was of course already created by first $\pi/2$ pulse, and then modified via the collisional effects.

8.4 Gross et al. (2011) [39]

• Stated emphasis - continuous variable entangled *twin-atom states*.

• System - Rb^{87} in several hyperfine states.

• Independent BECs of Rb^{87} trapped in separate wells in a optical lattice.

• Number of atoms macroscopic, ca few 100 in each well.

• Spin dynamics in Zeeman hyperfine states $(2,0)$, $(1,\pm1)$.

• Initially have BEC in $(2,0)$ hyperfine state - acts as pump mode.

• Spin conserving collisional coupling to $(1,\pm1)$ hyperfine states - which act as the two mode system.

• One boson created in each of $(1, \pm 1)$ hyperfine states with two bosons lost from $(2, 0)$ hyperfine state due spin conserving collisions.

• OPA type situation associated with spin changing collisions with $(1, \pm 1)$ hyperfine states acting as idler, signal modes .

• Mean and variance of population difference between $(1, +1)$ and $(1, -1)$ hyperfine states measured. Total population also measured.

• Entanglement test is that if the variance in population difference is small, but that in the total boson number is large then the state is entangled (see (73), (74)).

• Measurements (see Fig 1c in [39]) show noise in population difference is small, but that in the total boson number is large.

• Further entanglement test is that if there is two mode quadrature squeezing then the state is entangled.

• Comment - Number squeezing and two mode quadrature squeezing demonstrated and entanglement confirmed.

9 Discussion and Summary of Key Results

The two accompanying papers are concerned with mode entanglement for systems of identical massive bosons and the relationship to spin squeezing and other quantum correlation effects. These bosons may be atoms or molecules as in cold quantum gases. The previous paper I dealt with the general features of quantum entanglement and its specific definition in the case of systems of identical bosons. In defining entanglement for systems of identical massive particles, it was concluded that the single particle states or modes are the most appropriate choice for sub-systems that are distinguishable, that the general quantum states must comply both with the symmetrisation principle and the super-selection rules (SSR) that forbid quantum superpositions of states with differing total particle number (global SSR compliance), and that in the separable states quantum superpositions of sub-system states with differing sub-system particle number (local SSR compliance) also do not occur [1]. The present paper II has examined possible tests for two mode entanglement based on the treatment of entanglement set out in paper I.

The present paper first defines *spin squeezing* in *two mode* systems for the original spin operators $\widehat{S}_x, \widehat{S}_y, \widehat{S}_z$, which are defined in terms of the original mode annihilation and creation operators $\widehat{a}, \widehat{b}$ and $\widehat{a}^\dagger, \widehat{b}^\dagger$. Spin squeezing for the *principal spin operators* $\widehat{J}_x, \widehat{J}_y, \widehat{J}_z$ for which the *covariance matrix* is diagonal, rather than via the original spin operators is then discussed. It is seen that the two sets of spin operators are related via a rotation operator and the principal spin operators are given in terms of *new mode operators* $\widehat{c}, \widehat{d}$ and $\widehat{c}^\dagger, \widehat{d}^\dagger$, with $\widehat{c}, \widehat{d}$ obtained as linear combinations of the original mode operators $\widehat{a}, \widehat{b}$ and hence defining two new modes. Finally, we consider spin squeezing in the context of *multi-mode* systems.

The consequence for the case of two mode systems of identical bosons of the present approach to defining entangled states is that spin squeezing in *any* of the spin operators $\widehat{S}_x$, $\widehat{S}_y$ or $\widehat{S}_z$ *requires* entanglement of the original modes $\widehat{a}, \widehat{b}$. Similarly, spin squeezing in *any* of the new spin operators $\widehat{J}_x$, $\widehat{J}_y$ or $\widehat{J}_z$ requires entanglement of the new modes $\widehat{c}, \widehat{d}$. The full proof of these results has been presented here. A typical and *simple spin squeezing* test for entanglement is $\left\langle \Delta\widehat{S}_x^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|/2$ or $\left\langle \Delta\widehat{S}_y^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|/2$. We also found a simple *Bloch vector* test $\left\langle \widehat{S}_x \right\rangle \neq 0$ or $\left\langle \widehat{S}_y \right\rangle \neq 0$. It was noted that though spin squeezing requires entanglement, the opposite is not the case and the *NOON* state provided an example of an entangled physical state that is not spin squeezed. Also, the *binomial state* provided an example of a state that is entangled and spin squeezed for one choice of mode sub-systems but is non-entangled and not spin squeezed for another choice. The *relative phase state* provided an example that is entangled for new modes $\widehat{c}, \widehat{d}$ and is highly spin squeezed in $\widehat{J}_y$ and very unsqueezed in $\widehat{J}_x$. We then showed that in certain *multi-mode* cases, spin squeezing in any spin component confirmed entanglement. In the multi-mode case this test applied in the bipartite case (*Case 1*) where the *two sub-systems*

each consisted of all the modes $\widehat{a}_i$ *or* all the modes $\widehat{b}_i$ or in the *single modes* case (*Case 2*)where there were $2n$ sub-systems consisting of all the modes $\widehat{a}_i$ *and* all the modes $\widehat{b}_i$. For the *mode pairs* case (*Case 3*) where there were n sub-systems consisting of all the pairs of modes $\widehat{a}_i$ and $\widehat{b}_i$, a spin squeezing entanglement test was found in the situation where for separable states each mode pair involved a *single boson*. The connection between spin squeezing and entanglement was regarded as well-known, but up to now the only existing proofs were based on non-entangled states that disregarded either the symmetrization principle or the sub-system super-selection rules, placing the connection between spin squeezing and entanglement on a somewhat shaky basis. On the other hand, the proof given here is based on a definition of non-entangled (and hence entangled) states that is compatible with both these requirements.

There are several papers that have obtained *different tests* for whether a state is entangled from those involving *spin operators*, the proofs often being based on a definition of non-entangled states that ignores the sub-system SSR. Hillery et al [23] obtained criteria of this type, such as the *spin variance* entanglement test $\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle < \frac{1}{2}\left\langle \widehat{N} \right\rangle$. The proof of this test has also been set out here, and the test is also seen to be valid if the non-entangled state definition is consistent with the SSR. The test $\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|$ suggested by the requirement that $\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq |\left\langle \widehat{S}_z \right\rangle|$ for non-entangled states - since both $\left\langle \Delta\widehat{S}_x^2 \right\rangle \geq |\left\langle \widehat{S}_z \right\rangle|/2$ and $\left\langle \Delta\widehat{S}_y^2 \right\rangle \geq |\left\langle \widehat{S}_z \right\rangle|/2$ is of no use, since $\left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq |\left\langle \widehat{S}_z \right\rangle|$ for all states. However as previously noted, showing that either $\left\langle \Delta\widehat{S}_x^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|/2$ or $\left\langle \Delta\widehat{S}_y^2 \right\rangle < |\left\langle \widehat{S}_z \right\rangle|/2$ - or the analogous tests for other pairs of spin operators - already provides a test for the entanglement of the original modes $\widehat{a},\widehat{b}$. This test is a different test for entanglement than that of Hillery et al [23]. In fact the case of the *relative phase eigenstate* is an example of an entangled state in which the simple spin squeezing test for entanglement *succeeds* whereas that of Hillery et al [23] *fails*. The consequences of applying both the simple spin squeezing and the Hillery spin operator test were examined with the aim of seeing whether the results could determine whether or not the local particle number SSR applied to separable states. The conclusion was negative as all outcomes were consistent with both possibilities. In addition, the Hillery spin variance test was also shown to apply to the multi-mode situation, in the Cases 1 and 2 described above, but did not apply in Case 3. Other entanglement tests of Benatti et al [30] involving variances of two mode spin operators were also found to apply for identical bosons.

The present paper also considered *correlation tests* for entanglement. Inequalities found by Hillery et al [33] for non-entangled states which also do not depend on whether non-entangled states satisfy the super-selection rule include $|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n \right\rangle|^2 \leq \left\langle (\widehat{a}^\dagger)^m(\widehat{a})^m\,(\widehat{b}^\dagger)^n(\widehat{b})^n \right\rangle$, giving a valid *strong correlation* test $|\left\langle (\widehat{a})^m\,(\widehat{b}^\dagger)^n \right\rangle|^2 > \left\langle (\widehat{a}^\dagger)^m(\widehat{a})^m\,(\widehat{b}^\dagger)^n(\widehat{b})^n \right\rangle$ for an entangled state. However, with

entanglement defined as in the present paper we have $|\left\langle (\widehat{a})^{m}\,(\widehat{b}^{\dagger})^{n}\right\rangle|^{2}=0$ for a non-entangled state, so we have also proved a *weak correlation* test for entanglement in the form $|\left\langle (\widehat{a})^{m}\,(\widehat{b}^{\dagger})^{n}\right\rangle|^{2}>0$. This test is less stringent than the strong correlation test of Hillery et al [33]., as $|\left\langle (\widehat{a})^{m}\,(\widehat{b}^{\dagger})^{n}\right\rangle|^{2}$ is then required to be larger. In all these cases For $n\neq m$ none of these cases are of interest since for global SSR compliant states $\left\langle (\widehat{a})^{m}\,(\widehat{b}^{\dagger})^{n}\right\rangle$ would be zero. In the cases where $n=m$ we show that *all* the correlation tests can be expressed in terms of *spin operators*, so they reduce to tests involving powers of spin operators. For the case $n=m=1$ the weak correlation test is the same as the Bloch vector test.

Work by other authors on bipartite entanglement tests has also been examined here. He et al [26], [24] considered a *four mode* system, with two modes localised in each well of a double well potential. If the two sub-systems A and B each consist of two modes - with $\widehat{a}_{1},\,\widehat{a}_{2}$ as sub-system A and $\widehat{b}_{1},\,\widehat{b}_{2}$ as sub-system B, then tests of bipartite entanglement of the two sub-systems of the Hillery [33] type $|\left\langle (\widehat{a}_{i})^{m}\,(\widehat{b}_{j}^{\dagger})^{n}\right\rangle|^{2}>\left\langle (\widehat{a}_{i}^{\dagger})^{m}(\widehat{a}_{i})^{m}\,(\widehat{b}_{j}^{\dagger})^{n}(\widehat{b}_{j})^{n}\right\rangle$ for any $i,\ j=1,\,2$ or involving local spin operators $|\left\langle \widehat{S}_{+}^{A}\,\widehat{S}_{-}^{B}\right\rangle|^{2}>\left\langle \widehat{S}_{+}^{A}\,\widehat{S}_{-}^{A}\,\widehat{S}_{+}^{B}\,\widehat{S}_{-}^{B}\right\rangle$ apply. Raymer et al [27] have also considered such a four mode system and derived bipartite entanglement tests such as $\left\langle \Delta(\widehat{S}_{x}^{A}\pm\widehat{S}_{x}^{B})^{2}\right\rangle+\left\langle \Delta(\widehat{S}_{y}^{A}\mp\widehat{S}_{y}^{B})^{2}\right\rangle<|\left\langle \widehat{S}_{z}\right\rangle|$ that involve local spin operators for the two sub-systems.

We also considered the work of Sorensen et al [13], who showed that spin squeezing is a test for a state being entangled, but defined non-entangled states for identical particle systems (such as BECs) in a form that is *inconsistent* with the symmetrisation principle - the sub-systems being regarded as individual identical particles. However, the treatment of Sorensen et al [13] can be modified to apply to a system of identical bosons if the particle index i is *re-interpreted* as specifying diffferent modes, for example modes localised on optical lattice sites $i=1,2,..,n$ or localised in momentum space. With two single particle states $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ with annihilation operators a_{i},b_{i} available on each site, there would then be $2n$ modes involved, but spin operators can still be defined. This is just a particular case of the multi-mode situation described above. If the definitions of non–entangled and entangled states in the present paper are applied, it can be shown that spin squeezing in either of the spin operators $\widehat{S}_{x}$ or $\widehat{S}_{y}$ requires entanglement of *all* the original modes $\widehat{a}_{i},\widehat{b}_{i}$ (Case 1, above). Alternatively, if the sub-systems are *pairs* of modes $\widehat{a}_{i},\widehat{b}_{i}$ *and* the sub-system density operators $\widehat{\rho}_{R}^{i}$ were restricted to states with exactly *one boson*, then it can be shown that spin squeezing in $\widehat{S}_{z}$ requires entanglement of all the pairs of modes (Case 3, above). With this restriction the pair of modes $\widehat{a}_{i},\,\widehat{b}_{i}$ behave like *distinguishable* two state particles, which was essentially the case that Sorensen et al [13] implicitly considered. This type of entanglement is a multi-mode entanglement of a special type - since the modes $\widehat{a}_{i},\,\widehat{b}_{i}$ may themselves be entangled there is an "entanglement of entanglement". So with

either of these revisions, the work of Sorensen et al [13] could be said to show that spin squeezing requires entanglement. However, neither of these revisions really deals with the case of entanglement in *two mode* systems, and here the proof given in this paper showing that spin squeezing in $\widehat{S}_z$ requires entanglement of the two modes provides the justification of this result *without* treating identical particles as distinguishable sub-systems. Sorensen and Molmer [31] have also deduced an inequality involving $\left\langle \Delta \widehat{S}_{x}^{2} \right\rangle$ and $|\left\langle \widehat{S}_{z} \right\rangle|$ for states where $\left\langle \widehat{S}_{x} \right\rangle = \left\langle \widehat{S}_{y} \right\rangle = 0$ based on just the Heisenberg uncertainty principle. This is useful in terms of confirming that states do exist that are spin squeezed still conform to this principle.

Entanglement tests involving quadrature variables have also been published, so we have also examined these. Duan et al [28], Toth et al [34] devised a test for entanglement based on the sum of the *quadrature variances* $\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle \geq 2$ for separable states, which involve quadrature components $\widehat{x}_A, \widehat{p}_A, \widehat{x}_B, \widehat{p}_B$ constructed from the original mode annihilation, creation operators for modes A, B. Their conclusion that if the quadrature variances sum is less than 2 then the state is entangled is valid both for the present definition of entanglement and for that in which the application of the super-selection rule is ignored. However, for quantum states for systems of identical bosons that are global SSR compliant $\left\langle \Delta(\widehat{x}_A \pm \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A \mp \widehat{p}_B)^2 \right\rangle = 2 + 2\left\langle \widehat{N} \right\rangle$ for *all* such states - both separable and entangled, and although this is consistent with [28], [34] we have concluded that the quadrature variance test can *never* confirm entanglement. A more general test [35] involving quadrature operators $\widehat{X}_A^{\theta}, \widehat{X}_B^{\theta}$ required showing that $\left\langle \widehat{X}_A^{\theta}\, \widehat{X}_B^{\phi} \right\rangle \neq 0$. This was shown to be equivalent to showing that $\left\langle \widehat{S}_x \right\rangle \neq 0$ or $\left\langle \widehat{S}_y \right\rangle \neq 0$, the *Bloch vector* or *weak correlation* test. A *two mode quadrature squeezing* test was also obtained, but found to be less useful than the *Bloch vector* test.

Overall then, all of the *entanglement tests* (spin squeezing and other) in the other papers discussed here are *still valid* when reconsidered in accord with the definition of entanglement based on the symmetrisation and super-selection rules, though in one case Sorensen et al [13] a re-definition of the sub-systems is required to satisfy the symmetrization principle. However, *further* tests for entanglement are obtained in the present paper based on non-entangled states that are consistent with the symmetrization and super-selection rules. In some cases they are less stringent - the correlation test in Eq.(165) being easier to satisfy than that of Hillery et al [33] in Eq. (168). The tests introduced here are certainly *different* to others previously discovered.

The theory for a simple *two mode interferometer* was then presented and it was shown that such an interferometer can be used to measure the mean values and covariance matrix for the spin operators involved in entanglement tests for the two mode bosonic system. The treatmebnt was also generalised to *multi-mode* interferometry. The interferometer involved a *pulsed classical field* characterised by a *phase* variable ϕ and an *area* variable $2s = \theta$ defined by the

time integral of the field amplitude, and leads to a coupling between the two modes. For simplicity the centre frequency was chosen to be *resonant* with the mode transition frequency. Measuring the mean and variance of the *population difference* between the two modes for the *output* state of the interferometer for various choices of ϕ and θ enabled the mean values and covariance matrix for the spin operators for the *input* quantum state of the two mode system to be determined. More complex interferometers were seen to involve combinations of simple interferometers separated by time intervals during which *free evolution* of the two mode system can occur, including the effect of *collisions*.

Experiments have been carried out demonstrating that spin squeezing occurs, which according to theory requires entanglement. An analysis of these experiments has been presented here. However, since no results for entanglement *measures* are presented or other *independent* tests for entanglement carried out, the entanglement presumably created in the experiments has not been independently confirmed.

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11 Appendix A - Spin Squeezing Test for Bipartite Multi-Mode Case

We now consider spin squeezing for the multi-mode spin operators given in Eqs. (3) and (6) in SubSection 2.2. We consider separable states for *Case 1*, the density operator being given in Eq. (62). In this *bipartite case* the two subsystems consist of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$. The development involves expressions such as $\left\langle \widehat{\Psi}_c(\mathbf{r}) \right\rangle_R^C = Tr_C(\widehat{\Psi}_c(\mathbf{r})\widehat{\rho}_R^C)$, $\left\langle \widehat{\Psi}_c^\dagger(\mathbf{r}) \right\rangle_R^C = Tr_C(\widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\rho}_R^C)$ and $\left\langle \widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c^\dagger(\mathbf{r}') \right\rangle_R^C = Tr_C(\widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c^\dagger(\mathbf{r}')\widehat{\rho}_R^C)$, $\left\langle \widehat{\Psi}_c(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}') \right\rangle_R^C = Tr_C(\widehat{\Psi}_c(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}')\widehat{\rho}_R^C)$, $\left\langle \widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}') \right\rangle_R^C = Tr_C(\widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}')\widehat{\rho}_R^C)$, where $C = A, B$.

Firstly, we have

$$\left\langle \widehat{S}_x \right\rangle_R = \frac{1}{2}\int d\mathbf{r}\, \left(\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}) \right\rangle_R^A \left\langle \widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B \right) = 0 \tag{252}$$

since from the local particle number SSR for sub-systems A and B we have $\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \right\rangle_R^B = \left\langle \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A = 0$. A similar result applies to $\left\langle \widehat{S}_y \right\rangle_R$ so it then follows that

$$\left\langle \widehat{S}_x \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0 \tag{253}$$

This immediately yields the *Bloch vector* entanglement test. It also leads to the *spin squeezing* in $\widehat{S}_z$ entanglement test, namely if $\widehat{S}_z$ is squeezed with respect to $\widehat{S}_x$ or $\widehat{S}_y$ (or vice versa), then the state must be entangled. The question then is: Does spin squeezing in $\widehat{S}_x$ with respect to $\widehat{S}_y$ (or vice versa) require the state to be entangled for the two n mode sub-systems A and B?

To obtain an inequality for the variance in $\widehat{S}_x$, we see that

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle_R &= \frac{1}{4}\iint d\mathbf{r}\, d\mathbf{r}' \times \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A \\ &\quad + \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a^\dagger(\mathbf{r}') \right\rangle_R^A + \left\langle \widehat{\Psi}_b(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A \\ &\quad + \left\langle \widehat{\Psi}_b(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a^\dagger(\mathbf{r}') \right\rangle_R^A \} \end{aligned} \tag{254}$$

From the local particle number SSR for sub-systems A and B we have $\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B = \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A = 0$, so the first and fourth terms are zero. Using the field operator commutation rules we then obtain

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle_R &= \frac{1}{2}\iint d\mathbf{r}\, d\mathbf{r}'\, \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \\ &\quad + \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \end{aligned} \tag{255}$$

so that

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle_R = \frac{1}{2}\iint d\mathbf{r}\, d\mathbf{r}' \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A + \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \tag{256}$$

Hence from (32)

$$\left\langle \Delta\widehat{S}_x^2 \right\rangle \geq \sum_R P_R \{ \frac{1}{2}\iint d\mathbf{r}\, d\mathbf{r}' \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A + \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \tag{257}$$

The same result applies to $\left\langle \Delta\widehat{S}_y^2 \right\rangle$.

Now we can easily show that

$$\left\langle \widehat{S}_z \right\rangle = \sum_R P_R \frac{1}{2}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B - \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \tag{258}$$

so that

$$\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \leq \sum_R P_R \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \tag{259}$$

as $\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B$ and $\left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A$ are real and positive.

Hence we find that

$$\begin{aligned}
&\left\langle \Delta\widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\
\geq\ & \sum_R P_R \frac{1}{2}\iint d\mathbf{r}\, d\mathbf{r}' \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A && (260) \\
=\ & \sum_R P_R \frac{1}{2}\iint d\mathbf{r}\, d\mathbf{r}'\, Tr_B\{\widehat{\Psi}_b(\mathbf{r}')\,\widehat{\rho}_R^B\ \widehat{\Psi}_b^\dagger(\mathbf{r})\}\, Tr_A\{\widehat{\Psi}_a(\mathbf{r})\,\widehat{\rho}_R^A\ \widehat{\Psi}_a^\dagger(\mathbf{r}')\} && (261) \\
=\ & \frac{1}{2}\iint d\mathbf{r}\, d\mathbf{r}'\, Tr\left\{ \widehat{\Psi}_a(\mathbf{r})\,\widehat{\Psi}_b(\mathbf{r}')\,\widehat{\rho}_{sep}\,\widehat{\Psi}_a^\dagger(\mathbf{r}')\,\widehat{\Psi}_b^\dagger(\mathbf{r}) \right\} && (262)
\end{aligned}$$

giving three forms that the inequality for $\left\langle \Delta\widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$ has to satisfy in the case of a separable state. The last form involves a double space integral of a quantum correlation function. Note the order of $\mathbf{r}$ and $\mathbf{r}'$. It is straightforward to show that the right side of the inequality is real, but to achieve an entanglement test involving spin squeezing for $\widehat{S}_x$ we need to show that it is non-negative. Identical inequalities can be found for $\left\langle \Delta\widehat{S}_y^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$.

11.0.1 Mode Expansions

If we use Eq.(5) to expand the field operators then using Eq.(261) we have

$$\begin{aligned}
& \left\langle \Delta \widehat{S}_x^2 \right\rangle - \frac{1}{2} | \left\langle \widehat{S}_z \right\rangle | \\
\geq & \sum_R P_R \frac{1}{2} \sum_{ij} \sum_{kl} \iint d\mathbf{r}\, d\mathbf{r}' \\
& \times \{ \phi_i(\mathbf{r}) \phi_j^*(\mathbf{r}') \phi_k(\mathbf{r}') \phi_l^*(\mathbf{r}) \} \\
& \times \left\{ Tr_A \{ \widehat{a}_i \, \widehat{\rho}_R^A \, \widehat{a}_j^\dagger \} \, Tr_B \{ \widehat{b}_k \, \widehat{\rho}_R^B \, \widehat{b}_l^\dagger \} \right\} \\
= & \sum_R P_R \frac{1}{2} \sum_{ij} \left\{ Tr_A \{ \widehat{a}_i \, \widehat{\rho}_R^A \, \widehat{a}_j^\dagger \} \, Tr_B \{ \widehat{b}_j \, \widehat{\rho}_R^B \, \widehat{b}_i^\dagger \} \right\} \\
= & \sum_R P_R \frac{1}{4} \sum_{ij} (A_{ij}^R \, B_{ji}^R + B_{ij}^R \, A_{ji}^R) \qquad (263) \\
= & \sum_R P_R \frac{1}{4} Tr \{ A^R B^R + B^R A^R \} \qquad (264)
\end{aligned}$$

where mode orthogonality has been used and we have introduced *matrices* A^R and B^R whose elements are

$$A_{ij}^R = Tr_A \{ \widehat{a}_i \, \widehat{\rho}_R^A \, \widehat{a}_j^\dagger \} \qquad B_{ji}^R = Tr_B \{ \widehat{b}_j \, \widehat{\rho}_R^B \, \widehat{b}_i^\dagger \} \qquad (265)$$

It is easy to show that $A_{ij}^R = (A_{ji}^R)^*$ and $B_{ij}^R = (B_{ji}^R)^*$ showing that the matrices A^R and B^R are *Hermitian*, as is $A^R B^R + B^R A^R$. The quantity $\sum_{ij} (A_{ij}^R \, B_{ji}^R + B_{ij}^R \, A_{ji}^R)$ is *real*. The question is: Is it also *positive* ?

For the simple case where there is only *one* spatial mode for each component the right side of the inequality is just equal to $\sum_R P_R \frac{1}{2} \left\{ Tr_A \{ \widehat{a} \widehat{\rho}_R^A \, \widehat{a}^\dagger \} \, Tr_B \{ \widehat{b} \widehat{\rho}_R^B \, \widehat{b}^\dagger \} \right\} = \sum_R P_R \frac{1}{2} N_R^A \, N_R^B$, where N_R^A and N_R^B give the mean numbers of bosons in sub-systems A and B for the states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$. The right side of the inequality is positive, showing that the separable state is not spin squeezed for $\widehat{S}_x$ with respect to $\widehat{S}_y$ (or vice versa), leading as before to the test that such spin squeezing requires entanglement.

11.0.2 Positive Definiteness

For the multi-mode case we now take into account that the sub-system density operators $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are *positive-definite*. Their eigenvalues π_λ^{AR} and π_μ^{BR} are real and non-negative as well as summing to unity, and we can write the density operators in terms of their orthonormal eigenvectors $|AR, \lambda\rangle$ and $|BR, \mu\rangle$ as

$$\widehat{\rho}_R^A = \sum_\lambda \pi_\lambda^{AR} \, |AR, \lambda\rangle \, \langle AR, \lambda| \qquad \widehat{\rho}_R^B = \sum_\mu \pi_\mu^{BR} \, |BR, \mu\rangle \, \langle BR, \mu| \qquad (266)$$

Then from (265)

$$A^R_{ij} = \sum_\lambda \pi^{AR}_\lambda \langle AR,\lambda|\ \widehat{a}^\dagger_j \widehat{a}_i\ |AR,\lambda\rangle \qquad B^R_{ji} = \sum_\mu \pi^{BR}_\mu \langle BR,\mu|\ \widehat{b}^\dagger_i \widehat{b}_j\ |BR,\mu\rangle \tag{267}$$

Consider a $1 \times n$ row matrix $\xi^\dagger = \{\xi^*_1, \xi^*_2, .., \xi^*_n\ \}$

$$\begin{aligned}
\xi^\dagger A^R \xi &= \sum_{ij} \xi^*_i\, A^R_{ij}\, \xi_j \\
&= \sum_\lambda \pi^{AR}_\lambda \sum_{ij} \xi^*_i\ \langle AR,\lambda|\ \widehat{a}^\dagger_j \widehat{a}_i\ |AR,\lambda\rangle\ \xi_j \\
&= \sum_\lambda \pi^{AR}_\lambda\ \langle AR,\lambda|\ \widehat{\Omega}^\dagger_A\, \widehat{\Omega}_A\ |AR,\lambda\rangle
\end{aligned} \tag{268}$$

where we have introduced the operator $\widehat{\Omega}_A = \sum_i \xi^*_i\, \widehat{a}_i$. Since $\xi^\dagger A^R \xi$ is always non-negative for all ξ, this shows that A^R is a *positive definite* matrix. Similarly, considering a $1\times n$ row matrix $\eta^\dagger = \{\eta^*_1, \eta^*_2, .., \eta^*_n\ \}$ and introducing the operator $\widehat{\Omega}_B = \sum_i \eta_i\, \widehat{b}_i$ we find that

$$\eta^\dagger B^R \eta = \sum_\mu \pi^{BR}_\mu\ \langle BR,\mu|\ \widehat{\Omega}^\dagger_B\, \widehat{\Omega}_B\ |BR,\mu\rangle \tag{269}$$

which is also always non-negative, showing that B^R is also a *positive definite* matrix.

We can then express the positive definite Hermitian matrices A^R and B^R in terms of their normalised column eigenvectors θ^A_α and ζ^B_β respectively, where the corresponding real, positive eigenvalues are ν_α and σ_β Thus we have (for ease of notation R will be left understood)

$$\begin{aligned}
A^R\, \theta^A_\alpha &= \nu_\alpha \theta^A_\alpha \qquad (\theta^A_\alpha)^\dagger \theta^A_\gamma = \delta_{\alpha\gamma} \qquad A^R\ = \sum_\alpha \nu_\alpha \theta^A_\alpha\, (\theta^A_\alpha)^\dagger \\
B^R\, \zeta^B_\beta &= \sigma_\beta \zeta^B_\beta \qquad (\zeta^B_\beta)^\dagger \zeta^B_\epsilon = \delta_{\beta\epsilon} \qquad B^R\ = \sum_\beta \sigma_\beta \zeta^B_\beta\, (\zeta^B_\beta)^\dagger
\end{aligned} \tag{270}$$

Then

$$\begin{aligned}
& Tr\{A^R B^R + B^R A^R\} \\
=\ & Tr\{\sum_\alpha \sum_\beta \nu_\alpha \sigma_\beta\, \theta^A_\alpha\, (\theta^A_\alpha)^\dagger \zeta^B_\beta\, (\zeta^B_\beta)^\dagger\} \\
& + Tr\{\sum_\alpha \sum_\beta \nu_\alpha \sigma_\beta\, \zeta^B_\beta\, (\zeta^B_\beta)^\dagger\, \theta^A_\alpha\, (\theta^A_\alpha)^\dagger\} \\
=\ & \sum_\alpha \sum_\beta \nu_\alpha \sigma_\beta\, [(\theta^A_\alpha)^\dagger \zeta^B_\beta]\ [(\zeta^B_\beta)^\dagger \theta^A_\alpha] \\
& + \sum_\alpha \sum_\beta \nu_\alpha \sigma_\beta\, [(\zeta^B_\beta)^\dagger \theta^A_\alpha]\ [(\theta^A_\alpha)^\dagger \zeta^B_\beta] \\
=\ & 2 \sum_\alpha \sum_\beta \nu_\alpha \sigma_\beta\, |\, [(\theta^A_\alpha)^\dagger \zeta^B_\beta]\, |^2
\end{aligned} \tag{271}$$

Hence we have using (264)

$$
\begin{aligned}
&\left\langle \Delta \widehat{S}_x^2 \right\rangle - \frac{1}{2} |\left\langle \widehat{S}_z \right\rangle | \\
\geq \quad &\sum_R P_R \frac{1}{2} \sum_\alpha \sum_\beta \nu_\alpha \sigma_\beta \, | \, [(\theta_\alpha^A)^\dagger \zeta_\beta^B] \, |^2
\end{aligned} \tag{272}
$$

where the right side of the inequality is non-negative. The same result applies to $\left\langle \Delta \widehat{S}_y^2 \right\rangle - \frac{1}{2} |\left\langle \widehat{S}_z \right\rangle |$. Thus separable states are *not* spin squeezed in $\widehat{S}_x$ or in $\widehat{S}_y$.

Thus we have established the *spin squeezing* test for the multi–mode Case 1 - states that are spin squeezed in $\widehat{S}_x$ compared to $\widehat{S}_y$.(or vice versa) must be entangled states for the two subsystems consisting of all modes $\widehat{a}_i$ and all modes $\widehat{b}_i$.

For the other spin components, the Bloch vector result in (253) that $\left\langle \widehat{S}_x \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0$ for *separable* states enables us to show that if $\widehat{S}_z$ is squeezed compared to $\widehat{S}_x$.(or vice versa) or if $\widehat{S}_z$ is squeezed compared to $\widehat{S}_y$.(or vice versa) then the state must be entangled. Thus spin squeezing in *any* spin component requires the state to be entangled, justt as for the two mode case.

12 Appendix B - Spin Squeezing Tests for Other Multi-Mode Cases

12.1 Single Mode Sub-Systems

We now consider spin squeezing for the multi-mode spin operators given in Eqs. (3) and (6) in SubSection 2.2. We consider separable states for *Case 2*, the density operator being given in Eq. (63). In this *single mode sub-system case* there are $2n$ subsystems consist of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$.

This case is that involved in the modified approach to Sorensen et al and we will see that it leads to a useful inequality for $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle$ or.$\left\langle \Delta \widehat{S}_y^{\,2} \right\rangle$ that applies when non-entangled states are those when *all* the separate modes $\widehat{a}_i$ and $\widehat{b}_i$ are the sub-systems . We will follow the approach used for the simple two mode case in Section 3.

Firstly, the *variance* for a Hermitian operator $\widehat{\Omega}$ in a mixed state

$$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R \tag{273}$$

is always greater than or equal to the the average of the variances for the separate components

$$\left\langle \Delta \widehat{\Omega}^{\,2} \right\rangle \geq \sum_R P_R \left\langle \Delta \widehat{\Omega}^2 \right\rangle_R \tag{274}$$

where $\left\langle \Delta \widehat{\Omega}^{\,2} \right\rangle = Tr(\widehat{\rho}\, \Delta \widehat{\Omega}^{\,2})$ with $\Delta \widehat{\Omega} = \widehat{\Omega} - \left\langle \widehat{\Omega} \right\rangle$ and $\left\langle \Delta \widehat{\Omega}^{\,2} \right\rangle_R = Tr(\widehat{\rho}_R\, \Delta \widehat{\Omega}_R^{\,2})$ with $\Delta \widehat{\Omega}_R = \widehat{\Omega} - \left\langle \widehat{\Omega} \right\rangle_R$. The proof is straight-forward and given in Ref. [20].

Next we calculate $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle_R$, $\left\langle \Delta \widehat{S}_y^{\,2} \right\rangle_R$ and $\left\langle \widehat{S}_x \right\rangle_R$, $\left\langle \widehat{S}_y \right\rangle_R$, $\left\langle \widehat{S}_z \right\rangle_R$ for the case where

$$\widehat{\rho} = \sum_R P_R \left(\widehat{\rho}_R^{a\,1} \otimes \widehat{\rho}_R^{b\,1} \right) \otimes \left(\widehat{\rho}_R^{a\,2} \otimes \widehat{\rho}_R^{b\,2} \right) \otimes \left(\widehat{\rho}_R^{a\,3} \otimes \widehat{\rho}_R^{b\,3} \right) \otimes .... \tag{275}$$

as is required for a *general non-entangled* state *all* $2n$ modes. This situation is that of Choice 2 for the sub-systems, as described in SubSection 3.3. As the density operators for the individual modes must represent possible physical states for such modes, so super-selection rule for atom number applies and we have

$$\begin{aligned}
\langle (\widehat{a}_i)^p \rangle_{a\,i} &= Tr(\widehat{\rho}_R^{a\,i} (\widehat{a}_i)^p) = 0 \qquad \left\langle (\widehat{a}_i^\dagger)^p \right\rangle_{a\,i} = Tr(\widehat{\rho}_R^{a\,i} (\widehat{a}_i^\dagger)^p) = 0 \\
\left\langle (\widehat{b}_i)^m \right\rangle_{b\,i} &= Tr(\widehat{\rho}_R^{b\,i} (\widehat{b}_i)^m) = 0 \qquad \left\langle (\widehat{b}_i^\dagger)^m \right\rangle_{b\,i} = Tr(\widehat{\rho}_R^{b\,i} (\widehat{b}_i^\dagger)^m) = 0
\end{aligned} \tag{276}$$

The Schwinger spin operators are

$$\begin{aligned}
\widehat{S}_x &= \sum_i (\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i)/2 = \sum_i \widehat{S}_x^i \\
\widehat{S}_y &= \sum_i (\widehat{b}_i^\dagger \widehat{a}_i - \widehat{a}_i^\dagger \widehat{b}_i)/2i = \sum_i \widehat{S}_y^i \\
\widehat{S}_z &= \sum_i (\widehat{b}_i^\dagger \widehat{b}_i - \widehat{a}_i^\dagger \widehat{a}_i)/2 = \sum_i \widehat{S}_z^i
\end{aligned} \tag{277}$$

where $\widehat{a}_i$, $\widehat{b}_i$ and $\widehat{a}_i^\dagger$, $\widehat{b}_i^\dagger$ respectively are mode annihilation, creation operators. Note that this expression for the spin operators is the same as (6) for the multi-mode case treated in SubSection 2.2. From Eqs. (277) we find that

$$\widehat{S}_x^{\,2} = \sum_i (\widehat{S}_x^i)^2 + \sum_{i\neq j} \widehat{S}_x^i \widehat{S}_x^j \tag{278}$$

so that on taking the trace with $\widehat{\rho}_R$ and using Eqs. (275) we get after applying the commutation rules $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$)

$$\left\langle \widehat{S}_x^{\,2} \right\rangle_R = \sum_i \left\langle (\widehat{S}_x^i)^2 \right\rangle_R + \sum_{i\neq j} \left\langle \widehat{S}_x^i \right\rangle_R \left\langle \widehat{S}_x^j \right\rangle_R \tag{279}$$

As we also have

$$\left\langle \widehat{S}_x \right\rangle_R = \sum_i \left\langle \widehat{S}_x^i \right\rangle_R \qquad \left\langle \widehat{S}_x \right\rangle_R^2 = \sum_i \left\langle \widehat{S}_x^i \right\rangle_R^2 + \sum_{i\neq j} \left\langle \widehat{S}_x^i \right\rangle_R \left\langle \widehat{S}_x^j \right\rangle_R \tag{280}$$

using Eqs. (275) and we see finally that the variance $\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle_R$ is

$$\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle_R = \sum_i \left\langle (\widehat{S}_x^i)^2 \right\rangle_R - \sum_i \left\langle \widehat{S}_x^i \right\rangle_R^2 \tag{281}$$

all the terms with $i \neq j$ cancelling out. and therefore from Eq. (274)

$$\left\langle \Delta\widehat{S}_x^{\,2} \right\rangle \geq \sum_R P_R \sum_i \left(\left\langle (\widehat{S}_x^i)^2 \right\rangle_R - \left\langle \widehat{S}_x^i \right\rangle_R^2 \right) \tag{282}$$

An analogous result applies for $\left\langle \Delta\widehat{S}_y^{\,2} \right\rangle$.

But using (276)

$$\begin{aligned}
(\widehat{S}_x^i)^2 &= \frac{1}{4}(\widehat{b}_i^\dagger \widehat{a}_i \widehat{b}_i^\dagger \widehat{a}_i + \widehat{b}_i^\dagger \widehat{a}_i \widehat{a}_i^\dagger \widehat{b}_i + \widehat{a}_i^\dagger \widehat{b}_i \widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i \widehat{a}_i^\dagger \widehat{b}_i) \\
\left\langle (\widehat{S}_x^i)^2 \right\rangle_R &= \frac{1}{4}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R) + \frac{1}{2}(\left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R)
\end{aligned} \tag{283}$$

and

$$\left\langle \widehat{S}_x^i \right\rangle_R = 0 \tag{284}$$

It then follows that

$$\left\langle \widehat{S}_x \right\rangle = \sum_R P_R \left\langle \widehat{S}_x \right\rangle_R = 0 \qquad \left\langle \widehat{S}_y \right\rangle = \sum_R P_R \left\langle \widehat{S}_y \right\rangle_R = 0 \tag{285}$$

so that

$$\left\langle \Delta \widehat{S}_x^2 \right\rangle \geq \sum_R P_R \sum_i \left(\frac{1}{4}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R) + \frac{1}{2}(\left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R) \right) \tag{286}$$

The same result applies for $\left\langle \Delta \widehat{S}_y^2 \right\rangle$.

Now using (276)

$$\left\langle \widehat{S}_z^i \right\rangle_R = \frac{1}{2}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R - \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R)) \tag{287}$$

$$\begin{aligned} \left\langle \widehat{S}_z \right\rangle &= \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R \\ \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| &= \frac{1}{2}\sum_R P_R \,|\sum_i \frac{1}{2}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R - \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R))| \\ &\leq \sum_R P_R \frac{1}{4} \sum_i |(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R - \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R))| \\ &\leq \sum_R P_R \frac{1}{4} \sum_i (\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R)) \end{aligned} \tag{288}$$

and thus

$$\begin{aligned} & \left\langle \Delta \widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\ \geq \;& \sum_R P_R \sum_i \left(\frac{1}{4}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R) + \frac{1}{2}(\left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R) \right) \\ & - \sum_R P_R \frac{1}{4} \sum_i (\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R)) \\ = \;& \sum_R P_R \frac{1}{2} \sum_i (\left\langle (\widehat{a}^\dagger \widehat{a})_i \right\rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R) \\ \geq \;& 0 \end{aligned} \tag{289}$$

A similar proof shows that $\left\langle \Delta \widehat{S}_y^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \geq 0$ for the non-entangled state of all $2n$ modes.

This shows that for the general non-entangled state with all modes $\widehat{a}_i$ and $\widehat{b}_i$ as the sub-systems, the variances for two of the spin fluctuations $\left\langle \Delta \widehat{S}_x^2 \right\rangle$ and $\left\langle \Delta \widehat{S}_y^2 \right\rangle$ are both greater than $\frac{1}{2}|\left\langle \widehat{S}_z \right\rangle|$, and hence there is no spin squeezing for

$\widehat{S}_x$ or $\widehat{S}_y$. Note that as $|\left\langle \widehat{S}_y \right\rangle| = 0$, the quantity $\sqrt{\left(|\left\langle \widehat{S}_{\perp 1} \right\rangle|^2 + |\left\langle \widehat{S}_{\perp 2} \right\rangle|^2\right)}$ is the same as $|\left\langle \widehat{S}_z \right\rangle|$, so the alternative criterion in Eq. (16) is the same as that in Eq. (11) which is used here.

Hence we have shown that for a *non-entangled* physical state for all the $2n$ modes $\widehat{a}_i$ and $\widehat{b}_i$

$$\left\langle \Delta \widehat{S}_x^2 \right\rangle \geq \frac{1}{2} |\left\langle \widehat{S}_z \right\rangle| \quad \textit{and} \quad \left\langle \Delta \widehat{S}_y^2 \right\rangle \geq \frac{1}{2} |\left\langle \widehat{S}_z \right\rangle| \tag{290}$$

so that spin squeezing in either $\widehat{S}_x$ or $\widehat{S}_y$ requires entanglement.

From (285) we see that $\left\langle \widehat{S}_x \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0$ for the general separable state, showing there is a *Bloch vector test* for entanglement such that if either $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero, then the state must be entangled.

Finally, if there is spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_x$ or vice versa, or spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_y$ or vice versa, it follows that one of $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero. But as both these quantities are zero for a non-entangled state, if follows that spin squeezing in $\widehat{S}_z$ also requires entanglement.

Thus, spin squeezing in *any* spin operator $\widehat{S}_x, \widehat{S}_y$ or $\widehat{S}_z$ is a sufficiency test for entanglement of all the separate mode sub-systems.

12.2 Two Mode Sub-Systems

We now consider spin squeezing for the multi-mode spin operators given in Eqs. (3) and (6) in SubSection 2.2. We consider separable states for *Case 3*, the density operator being given in Eq. (64). In this *mode pair sub-system case* there are n subsystems consist of *all pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$.

This case is also involved in a modified approach to Sorensen et al and we show a useful inequality for $\left\langle \Delta \widehat{S}_z^2 \right\rangle$ applies when non-entangled states are those when the *pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$ are the separate sub-systems, but only in restricted situations. The pairs of modes corresponding to localised modes on different lattice sites or pairs of modes with the same momenta doe represent the closest way of simulating the approach used by Sorensen et al where identical particles i were regarded as the sub-systems.

Now the general non-entangled state will be

$$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R^1 \otimes \widehat{\rho}_R^2 \otimes \widehat{\rho}_R^3 \otimes ... \tag{291}$$

where the $\widehat{\rho}_R^i$ are now the density operators for sub-system i consisting of the pair of modes $\widehat{a}_i$ and $\widehat{b}_i$ (which are of the form given in Eq. (138)) and the conditions in Eq. (276) no longer apply. The Fock states are of the form $|N_{ia}\rangle \otimes |N_{ib}\rangle$ for the

pair of modes $\widehat{a}_i$ and $\widehat{b}_i$, and for this Fock state the total occupancy of the pair of modes is $N_i = N_{ia} + N_{ib}$. From the super-selection rule the density operator $\widehat{\rho}_R^i$ for the ith pair of modes $\widehat{a}_i$ and $\widehat{b}_i$ is diagonal in the total occupancy. For $N_i = 0$ there is one non zero matrix element $(\langle 0|_{ia} \otimes \langle 0|_{ib})\, \widehat{\rho}_R^i\, (|0\rangle_{ia} \otimes |0\rangle_{ib})$. For $N_i = 1$ there are four non zero matrix elements, which may be written

$$\begin{aligned}
(\langle 1|_{ia} \otimes \langle 0|_{ib})\, \widehat{\rho}_R^i\, (|1\rangle_{ia} \otimes |0\rangle_{ib}) &= \rho_{aa}^i \\
(\langle 1|_{ia} \otimes \langle 0|_{ib})\, \widehat{\rho}_R^i\, (|0\rangle_{ia} \otimes |1\rangle_{ib}) &= \rho_{ab}^i \\
(\langle 0|_{ia} \otimes \langle 1|_{ib})\, \widehat{\rho}_R^i\, (|1\rangle_{ia} \otimes |0\rangle_{ib}) &= \rho_{ba}^i \\
(\langle 0|_{ia} \otimes \langle 1|_{ib})\, \widehat{\rho}_R^i\, (|0\rangle_{ia} \otimes |1\rangle_{ib}) &= \rho_{bb}^i
\end{aligned} \tag{292}$$

For $N_i = 2$ there are nine non zero matrix element $(\langle 2|_{ia} \otimes \langle 0|_{ib})\, \widehat{\rho}_R^i\, (|2\rangle_{ia} \otimes |0\rangle_{ib})$, ..., $(\langle 0|_{ia} \otimes \langle 2|_{ib})\, \widehat{\rho}_R^i\, (|0\rangle_{ia} \otimes |2\rangle_{ib})$ and the number increases with N_i.

If we restrict ourselves to general entangled states for *one particle*, where $N_i = 1$ for all pairs of modes, then the density operator $\widehat{\rho}_R^i$ is of then form

$$\begin{aligned}
\widehat{\rho}_R^i &= \rho_{aa}^i(|1\rangle_{ia} \langle 1|_{ia} \otimes |0\rangle_{ib} \langle 0|_{ib}) + \rho_{ab}^i(|1\rangle_{ia} \langle 0|_{ia} \otimes |0\rangle_{ib} \langle 1|_{ib}) \\
&\quad + \rho_{ba}^i(|0\rangle_{ia} \langle 1|_{ia} \otimes |1\rangle_{ib} \langle 0|_{ib}) + \rho_{bb}^i(|0\rangle_{ia} \langle 0|_{ia} \otimes |1\rangle_{ib} \langle 1|_{ib})
\end{aligned} \tag{293}$$

In addition Hermitiancy, positivity, unit trace $Tr(\widehat{\rho}_R^i) = 1$ and $Tr(\widehat{\rho}_R^i)^2 \leq 1$ can be used as in Eq (327) to parameterise the matrix elements in (292).

$$\begin{aligned}
\rho_{aa}^i &= \sin^2 \alpha_i \qquad \rho_{bb}^i = \cos^2 \alpha_i \\
\rho_{ab}^i &= \sqrt{\sin^2 \alpha_i \, \cos^2 \alpha_i}\, \sin^2 \beta_i \, \exp(+i\phi_i) \qquad \rho_{ba}^i = \sqrt{\sin^2 \alpha_i \, \cos^2 \alpha_i}\, \sin^2 \beta_i \, \exp(-i\phi_i)
\end{aligned} \tag{294}$$

The expectation values for the spin operators $\widehat{S}_x^i$, $\widehat{S}_y^i$ and $\widehat{S}_z^i$ associated with the ith pair of modes are then

$$\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2}(\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i) \\
&= \frac{1}{2}\left(\rho_{ab}^i + \rho_{ba}^i\right) \\
\left\langle \widehat{S}_y^i \right\rangle_R &= \frac{1}{2i}\left(\rho_{ab}^i - \rho_{ba}^i\right) \\
\left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\left(\rho_{bb}^i - \rho_{aa}^i\right)
\end{aligned} \tag{295}$$

which are of exactly the same form as in Eq. (326) as in the Appendix 14 derivation of the original Sorensen et al [13] results based on treating identical particles as the sub-systems. The proof however is now different and rests on restricting the states $\widehat{\rho}_R^i$ to each containing exactly one boson.

The remainder of the proof is exactly the same as in Appendix 14 and we find that

$$\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right) \tag{296}$$

for non-entangled *pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$. Thus when the interpretation is changed so that are the separate sub-systems are these pairs of modes, it follows that spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_x$ or $\widehat{S}_y$ requires entanglement of all the mode pairs, but only if there is one particle in each mode pair.

In general, spin squeezing in either $\widehat{S}_x$ or $\widehat{S}_y$ is not linked to entanglement for Case 3 sub-systems, as has been pointed out in SubSection 3.3 by a counter-example involving the relative phase state. Also there is no Bloch vector entanglement test. For we have in general

$$\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2}(\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i) \\
\left\langle \widehat{S}_y^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2i}(\widehat{b}_i^\dagger \widehat{a}_i - \widehat{a}_i^\dagger \widehat{b}_i)
\end{aligned} \tag{297}$$

and the local particle number SSR does not require these quantities to be zero for sub-systems consisting of pairs of modes $\widehat{a}_i$ and $\widehat{b}_i$. Thus in general $\left\langle \widehat{S}_x \right\rangle$ and $\left\langle \widehat{S}_y \right\rangle$ can be non-zero for a separable state, so the Bloch vector entanglement test does not apply.

13 Appendix C - Hillery Spin Variance - Multi-Mode

13.1 Bipartite Case

We first consider *Case 1* where there are *two sub-systems* each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$. We use the results from (257) to find that for a separable state

$$\begin{aligned} & \left\langle \Delta \widehat{S}_x^{\,2} \right\rangle + \left\langle \Delta \widehat{S}_y^{\,2} \right\rangle \\ \geq \ & \sum_R P_R \{ \iint d\mathbf{r}\, d\mathbf{r'} \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \widehat{\Psi}_b(\mathbf{r'}) \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r'}) \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \\ & + \frac{1}{2} \int d\mathbf{r} \, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}) \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \end{aligned} \tag{298}$$

The same result would have occured if the local sub-system SSR had been disregarded, the terms such as $\frac{1}{4} \iint d\mathbf{r}\, d\mathbf{r'} \times \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \widehat{\Psi}_b^\dagger(\mathbf{r'}) \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r}) \widehat{\Psi}_a(\mathbf{r'}) \right\rangle_R^A$ cancelling out.

The mean number of bosons is obtained from (7) and hence

$$\frac{1}{2} \left\langle \widehat{N} \right\rangle = \frac{1}{2} \sum_R P_R \int d\mathbf{r} \left(\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}) \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \right) \tag{299}$$

Thus we have

$$\begin{aligned} & \left\langle \Delta \widehat{S}_x^{\,2} \right\rangle + \left\langle \Delta \widehat{S}_y^{\,2} \right\rangle - \frac{1}{2} \left\langle \widehat{N} \right\rangle \\ \geq \ & \sum_R P_R \iint d\mathbf{r}\, d\mathbf{r'} \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \widehat{\Psi}_b(\mathbf{r'}) \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r'}) \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \end{aligned} \tag{300}$$

Using the mode expansion (5) we then get

$$\begin{aligned} & \left\langle \Delta \widehat{S}_x^{\,2} \right\rangle + \left\langle \Delta \widehat{S}_y^{\,2} \right\rangle - \frac{1}{2} \left\langle \widehat{N} \right\rangle \\ \geq \ & \sum_R P_R \sum_{ij} \sum_{kl} \iint d\mathbf{r}\, d\mathbf{r'}\, \phi_i^*(\mathbf{r}) \phi_j(\mathbf{r'}) \phi_k^*(\mathbf{r'}) \phi_l(\mathbf{r}) \left\langle \widehat{b}_i^\dagger \widehat{b}_j \right\rangle_R^B \left\langle \widehat{a}_k^\dagger \widehat{a}_l \right\rangle_R^A \\ = \ & \sum_R P_R \sum_{ij} Tr_A\{ \widehat{a}_i \, \widehat{\rho}_R^A \, \widehat{a}_j^\dagger \} Tr_B \{ \widehat{b}_j \, \widehat{\rho}_R^B \, \widehat{b}_i^\dagger \} \end{aligned} \tag{301}$$

$$= \sum_R P_R \frac{1}{2} Tr(A^R B^R + B^R A^R) \tag{302}$$

after orthogonality is used and the matrix elements A_{ij}^R and B_{ji}^R are introduced from (265).

Since we have shown in Appendix 11 that the right side of the last inequality is always non-negative, the *Hillery spin variance* entanglement test follows that if

$$\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle + \left\langle \Delta \widehat{S}_y^{\,2} \right\rangle < \frac{1}{2} \left\langle \widehat{N} \right\rangle \tag{303}$$

then the quantum state must be an entangled state for the case of two sub-systems each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$.

13.2 Single Modes Case

We now consider separable states for *Case 2*, the density operator being given in Eq. (63). In this *single mode sub-system case* there are $2n$ subsystems consisting of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$. We use the results from (286) to find that for a separable state

$$\begin{aligned}&\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle \\ \geq\ &\sum_R P_R \sum_i \left(\frac{1}{2}(\left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R + \left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R) + (\left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R \left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R)\right)\end{aligned} \tag{304}$$

The same result would have occured if the local sub-system SSR had been disregarded, the terms such as $\frac{1}{4}\left\langle \widehat{b}_i^\dagger\widehat{b}_i^\dagger\right\rangle_{Ri}\langle\widehat{a}_i\widehat{a}_i\rangle_R$ cancelling out.

The mean number of bosons is obtained from (7)

$$\frac{1}{2}\left\langle \widehat{N}\right\rangle = \frac{1}{2}\sum_R P_R \sum_i(\left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R + \left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R) \tag{305}$$

Thus we have

$$\begin{aligned}&\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle - \frac{1}{2}\left\langle \widehat{N}\right\rangle \\ \geq\ &\sum_R P_R \sum_i \left(\left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R \left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R\right)\end{aligned} \tag{306}$$

which is always non-negative.

The *Hillery spin variance* entanglement test follows that if the inequality in (303) occurs then the quantum state must be an entangled state for the case of $2n$ sub-systems consisting of all the modes $\widehat{a}_i$ and all the modes $\widehat{b}_i$.

13.3 Two Modes Case

We now consider separable states for *Case 3*, the density operators being given in Eq. (64). In this *two mode sub-system case* there are n subsystems consisting of *all* mode pairs $\widehat{a}_i$ and $\widehat{b}_i$. We consider a *special* separable state with just one term where

$$\widehat{\rho}_{sep} = \widehat{\rho}^{ab(1)} \otimes \widehat{\rho}^{ab(2)} \otimes .. \otimes \widehat{\rho}^{ab(i)} .. \otimes \widehat{\rho}^{ab(n)} \tag{307}$$

We use the results from (281) to find that

$$\begin{aligned}&\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle \\ =\ &\sum_i \left(\left\langle (\Delta\widehat{S}_x^i)^2\right\rangle + \left\langle (\Delta\widehat{S}_y^i)^2\right\rangle\right)\end{aligned} \tag{308}$$

where $\Delta\widehat{S}_{\alpha}^{i}=\widehat{S}_{\alpha}^{i}-\left\langle \widehat{S}_{\alpha}^{i}\right\rangle_{R}$ for $\alpha=x,y$. This result did not depend on applying the local SSR.

Now suppose each of the two mode states $\widehat{\rho}^{ab(i)}$ is an entangled state of the modes $\widehat{a}_{i}$ and $\widehat{b}_{i}$ in which the Hillery spin variance test is satisfied. Then

$$\left\langle (\Delta\widehat{S}_{x}^{i})^{2}\right\rangle+\left\langle (\Delta\widehat{S}_{y}^{i})^{2}\right\rangle<\frac{1}{2}\left\langle \widehat{n}_{i}\right\rangle$$

Hence

$$\begin{aligned}
& \left\langle \Delta\widehat{S}_{x}^{\,2}\right\rangle+\left\langle \Delta\widehat{S}_{y}^{\,2}\right\rangle \\
< & \sum_{i}\frac{1}{2}\left\langle \widehat{N}_{i}\right\rangle \\
= & \frac{1}{2}\left\langle \widehat{N}\right\rangle
\end{aligned} \tag{309}$$

where $\widehat{N}=\sum_{i}\widehat{N}_{i}$ is the total number operator and $\widehat{N}_{i}=\widehat{b}_{i}^{\dagger}\widehat{b}_{i}+\widehat{a}_{i}^{\dagger}\widehat{a}_{i}$

Thus the Hillery spin variance test is satisfied even though the state (307) is separable, shoeing that the test cannot be applied for multi-mode Case 3.

14 Appendix D - Derivation of Sorensen et al Results

Sorensen et al [13] derive a number of ineqalities from which they deduce a further inequality for the spin squeezing parameter in the case of a non-entangled state. From this result they conclude that spin squeezing implies entanglement. The final inequality they obtain for a non-entangled state is

$$\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right) \tag{310}$$

Their approach is based on writing the density operator for a non-entangled state of N identical particles as in Eq. (130)

$$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R^1 \otimes \widehat{\rho}_R^2 \otimes \widehat{\rho}_R^3 \otimes ... = \sum_R P_R \, \widehat{\rho}_R \tag{311}$$

The spin operators are defined as

$$\begin{aligned}
\widehat{S}_x &= \sum_i \widehat{S}_x^i = \sum_i (|\phi_b(i)\rangle \langle \phi_a(i)| + |\phi_a(i)\rangle \langle \phi_b(i)|)/2 \\
\widehat{S}_y &= \sum_i \widehat{S}_y^i = \sum_i (|\phi_b(i)\rangle \langle \phi_a(i)| - |\phi_a(i)\rangle \langle \phi_b(i)|)/2i \\
\widehat{S}_z &= \sum_i \widehat{S}_z^i = \sum_i (|\phi_b(i)\rangle \langle \phi_b(i)| - |\phi_a(i)\rangle \langle \phi_a(i)|)/2
\end{aligned} \tag{312}$$

where the sum i is over the identical atoms and each atom is associated with two states $|\phi_a\rangle$ and $|\phi_b\rangle$. Clearly, the spin operators satisfy the standard commutation rules for agular momentum operators.

Sorensen et al [13] state that the variance for $\widehat{S}_z$ satisfies the result

$$\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle = \frac{N}{4} - \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 + \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 - \left\langle \widehat{S}_z \right\rangle^2 \tag{313}$$

To prove this we have

$$\begin{aligned}
\left\langle \widehat{S}_z^{\,2} \right\rangle &= \sum_R P_R \, Tr(\widehat{\rho}_R \sum_i \sum_j \widehat{S}_z^i \widehat{S}_z^j) \\
&= \sum_R P_R \left(\sum_i \left\langle \left(\widehat{S}_z^i \right)^2 \right\rangle_R + \sum_{i \neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right) \\
&= \frac{N}{4} + \sum_R P_R \left(\sum_{i \neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right)
\end{aligned} \tag{314}$$

where we have used

$$\begin{aligned}
\left(\widehat{S}_z^i\right)^2 &= \frac{1}{4}(|\phi_b(i)\rangle\langle\phi_b(i)| - |\phi_a(i)\rangle\langle\phi_a(i)|)^2 \\
&= \frac{1}{4}(|\phi_b(i)\rangle\langle\phi_b(i)|\phi_b(i)\rangle\langle\phi_b(i)| - (|\phi_b(i)\rangle\langle\phi_b(i)|\phi_a(i)\rangle\langle\phi_a(i)|) \\
&\quad + \frac{1}{4}(-(|\phi_a(i)\rangle\langle\phi_a(i)|\phi_b(i)\rangle\langle\phi_b(i)| + (|\phi_a(i)\rangle\langle\phi_a(i)|\phi_a(i)\rangle\langle\phi_a(i)|) \\
&= \frac{1}{4}((|\phi_b(i)\rangle\langle\phi_b(i)| + (|\phi_a(i)\rangle\langle\phi_a(i)|) \\
&= \frac{1}{4}\widehat{1}_i
\end{aligned} \tag{315}$$

a result based on the orthogonality, normalisation and completeness of the states $|\phi_a(i)\rangle, |\phi_b(i)\rangle$. Also

$$\begin{aligned}
\left\langle \widehat{S}_z \right\rangle_R &= Tr(\widehat{\rho}_R \sum_i \widehat{S}_z^i) \\
&= \sum_i \left\langle \widehat{S}_z^i \right\rangle_R \\
\sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 &= \sum_R P_R \left(\sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 + \sum_{i\neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right)
\end{aligned} \tag{316}$$

so eliminating the term $\sum_R P_R \left(\sum_{i\neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right)$ gives the required expression for $\left\langle \Delta\widehat{S}_z^{\,2} \right\rangle = \left\langle \widehat{S}_z^{\,2} \right\rangle - \left\langle \widehat{S}_z \right\rangle^2$.

Next, Sorensen et al [13] state that

$$\left\langle \widehat{S}_x \right\rangle^2 \leq N \sum_R P_R \sum_i \left\langle \widehat{S}_x^i \right\rangle_R^2 \qquad \left\langle \widehat{S}_y \right\rangle^2 \leq N \sum_R P_R \sum_i |\left\langle \widehat{S}_y^i \right\rangle_R|^2 \tag{317}$$

To prove this we have

$$\begin{aligned}
\left\langle \widehat{S}_x \right\rangle &= \sum_R P_R \, Tr(\widehat{\rho}_R \sum_i \widehat{S}_x^i) \\
&= \sum_R P_R \sum_i \left\langle \widehat{S}_x^i \right\rangle_R \\
|\left\langle \widehat{S}_x \right\rangle| &\leq \sum_R P_R \sum_i |\left\langle \widehat{S}_x^i \right\rangle_R|
\end{aligned} \tag{318}$$

since the modulus of a sum is less than or equal to the sum of the moduli. Now

$$\begin{aligned}
\left\langle \widehat{S}_x \right\rangle^2 &= |\left\langle \widehat{S}_x \right\rangle|^2 \leq \left(\sum_R P_R \sum_i |\left\langle \widehat{S}_x^i \right\rangle_R| \right)^2 \\
&\leq \sum_R P_R \left(\sum_i |\left\langle \widehat{S}_x^i \right\rangle_R| \right)^2
\end{aligned} \tag{319}$$

using the general result that $\left(\sum\limits_R P_R\sqrt{C_R}\right)^2 \leq \sum\limits_R P_R\, C_R$, where $\sum\limits_R P_R = 1$ with here $\sqrt{C_R} = \sum\limits_i |\left\langle \widehat{S}_x^i \right\rangle_R|$. Next consider

$$\begin{aligned}
y &= N\sum_i |\left\langle \widehat{S}_x^i \right\rangle_R|^2 \\
z &= \left(\sum_i |\left\langle \widehat{S}_x^i \right\rangle_R|\right)^2 = \left(\sum_i |\left\langle \widehat{S}_x^i \right\rangle_R|\right)^2 \\
y - z &= \sum_{i<j}(|\left\langle \widehat{S}_x^i \right\rangle_R| - |\left\langle \widehat{S}_x^j \right\rangle_R|)^2 \geq 0
\end{aligned} \tag{320}$$

so that

$$\left\langle \widehat{S}_{\,x} \right\rangle^2 \leq N\sum_R P_R\,\sum_i |\left\langle \widehat{S}_x^i \right\rangle_R|^2 \qquad \left\langle \widehat{S}_{\,y} \right\rangle^2 \leq N\sum_R P_R\,\sum_i |\left\langle \widehat{S}_y^i \right\rangle_R|^2 \tag{321}$$

which is the required result. The inequality for $\left\langle \widehat{S}_{\,y} \right\rangle^2$ is proved similarly.

Another inequality is stated [13] for $\left\langle \widehat{S}_{\,z} \right\rangle^2$. This is

$$\left\langle \widehat{S}_{\,z} \right\rangle^2 \leq \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 \tag{322}$$

To show this we have

$$\begin{aligned}
\left\langle \widehat{S}_{\,z} \right\rangle &= \sum_R P_R\, Tr(\widehat{\rho}_R \sum_i \widehat{S}_z^i) \\
&= \sum_R P_R\, \sum_i \left\langle \widehat{S}_z^i \right\rangle_R \\
&= \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R \\
|\left\langle \widehat{S}_{\,z} \right\rangle| &\leq \sum_R P_R\, |\left\langle \widehat{S}_z \right\rangle_R|
\end{aligned} \tag{323}$$

so that

$$\begin{aligned}
\left\langle \widehat{S}_{\,z} \right\rangle^2 &= |\left\langle \widehat{S}_{\,z} \right\rangle|^2 \leq \left(\sum_R P_R\, |\left\langle \widehat{S}_z \right\rangle_R|\right)^2 \\
&\leq \sum_R P_R\, |\left\langle \widehat{S}_z \right\rangle_R|^2 \\
&= \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2
\end{aligned} \tag{324}$$

using the general result that $\left(\sum_R P_R \sqrt{C_R}\right)^2 \leq \sum_R P_R\, C_R$, where $\sum_R P_R = 1$ with here $\sqrt{C_R} = |\left\langle \widehat{S}_z \right\rangle_R|$.

Finally, we find that

$$
\begin{aligned}
\sum_R P_R \sum_i \left(\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 \right) &\leq \frac{1}{4}N \\
-\sum_R P_R \sum_i \left(\left\langle \widehat{S}_z^i \right\rangle_R^2 \right) &\geq -\frac{1}{4}N + \sum_R P_R \sum_i \left(\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 \right)
\end{aligned}
\tag{325}
$$

To show this we use the properties of the density operator $\widehat{\rho}_R^i$ for the ith particle of Hermitiancy, positiveness, unit trace $Tr(\widehat{\rho}_R^i) = 1$ and $Tr(\widehat{\rho}_R^i)^2 \leq 1$. In terms of matrix elements of the density operator $\widehat{\rho}_R^i$ between the two states $|\phi_a(i)\rangle$, $|\phi_b(i)\rangle$ the quantities $\left\langle \widehat{S}_x^i \right\rangle_R$, $\left\langle \widehat{S}_y^i \right\rangle_R$ and $\left\langle \widehat{S}_z^i \right\rangle_R$ are

$$
\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2}(|\phi_b(i)\rangle \langle \phi_a(i)| + |\phi_a(i)\rangle \langle \phi_b(i)|)) \\
&= \frac{1}{2}\left(\rho_{ab}^i + \rho_{ba}^i\right) \\
\left\langle \widehat{S}_y^i \right\rangle_R &= \frac{1}{2i}\left(\rho_{ab}^i - \rho_{ba}^i\right) \\
\left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\left(\rho_{bb}^i - \rho_{aa}^i\right)
\end{aligned}
\tag{326}
$$

where $\rho_{cd}^i = \langle \phi_c(i)| \widehat{\rho}_R^i \, |\phi_d(i)\rangle$. The Hermitiancy and positiveness of $\widehat{\rho}_R^i$ show that ρ_{bb}^i and ρ_{aa}^i are real and positive, $\rho_{ab}^i = (\rho_{ba}^i)^*$ and $\rho_{aa}^i \rho_{bb}^i - |\rho_{ab}^i|^2 \geq 0$. The condition $Tr(\widehat{\rho}_R^i) = 1$ leads to $\rho_{aa}^i + \rho_{bb}^i = 1$, from which $Tr(\widehat{\rho}_R^i)^2 \leq 1$ follows using the previous positivity results. Taken together these conditions lead to the following useful parametrisation of the density matrix elements

$$
\begin{aligned}
\rho_{aa}^i &= \sin^2\alpha_i \qquad \rho_{bb}^i = \cos^2\alpha_i \\
\rho_{ab}^i &= \sqrt{\sin^2\alpha_i \, \cos^2\alpha_i}\, \sin^2\beta_i \, \exp(+i\phi_i) \qquad \rho_{ba}^i = \sqrt{\sin^2\alpha_i \, \cos^2\alpha_i}\, \sin^2\beta_i \, \exp(-i\phi_i)
\end{aligned}
\tag{327}
$$

where α_i, β_i and ϕ_i are real. In terms of these quantities we then have

$$
\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= \frac{1}{2}\sin 2\alpha_i \, \sin^2\beta_i \, \cos\phi_i \\
\left\langle \widehat{S}_y^i \right\rangle_R &= \frac{1}{2}\sin 2\alpha_i \, \sin^2\beta_i \, \sin\phi_i \\
\left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\cos 2\alpha_i
\end{aligned}
\tag{328}
$$

It is then easy to show that

$$\begin{aligned}\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 &= \frac{1}{4} - \frac{1}{4}\sin^2 2\alpha_i \,(1 - \sin^4 \beta_i\,) \\ &\leq \frac{1}{4} \end{aligned} \tag{329}$$

and the final inequality (325) then follows by taking the sum over particles i and then using $\sum_R P_R = 1$. If only the Schwarz inequality is used instead of the more detailed consequences of Hermtiancy, positiveness etc it can be shown that $\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 \leq \frac{3}{4}$, which though correct is not useful.

Combining the inequalities in Eqs. (317), (322) and (325) into Eq. (313) shows that

$$\begin{aligned}\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle &= \frac{N}{4} - \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 + \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 - \left\langle \widehat{S}_z \right\rangle^2 \\ &\geq \frac{N}{4} - \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 \\ &\geq \frac{N}{4} - \frac{1}{4}N + \sum_R P_R \sum_i \left(\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 \right) \\ &\geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right) \end{aligned} \tag{330}$$

for the case of a non-entangled state. This result is that in Sorensen et al.[13].

15 Appendix E - Heisenberg Uncertainty Principle Results

Here we derive the results in SubSection 4.6 leading to inequalities for the variance $\left\langle \Delta\widehat{J}_x^2 \right\rangle$ considered as a function of $|\left\langle \widehat{J}_z \right\rangle|$ for states where the spin operators are chosen such that $\left\langle \widehat{J}_x \right\rangle = \left\langle \widehat{J}_y \right\rangle = 0$.

From the Schwarz inequality $\left\langle \widehat{J}_z \right\rangle^2 \leq \left\langle \widehat{J}_z^2 \right\rangle$ so that

$$\left\langle \widehat{J}_x^2 \right\rangle + \left\langle \widehat{J}_y^2 \right\rangle + \left\langle \widehat{J}_z \right\rangle^2 \leq \left\langle \widehat{J}_x^2 \right\rangle + \left\langle \widehat{J}_y^2 \right\rangle + \left\langle \widehat{J}_z^2 \right\rangle = J(J+1) \tag{331}$$

giving Eq. (157). Subtracting $\left\langle \widehat{J}_x \right\rangle^2 = \left\langle \widehat{J}_y \right\rangle^2 = 0$ from each side gives

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle + \left\langle \Delta\widehat{J}_y^2 \right\rangle + \left\langle \widehat{J}_z \right\rangle^2 \leq J(J+1) \tag{332}$$

Substituting for $\left\langle \Delta\widehat{J}_y^2 \right\rangle$ from the Heisenberg uncertainty principle result in Eq. (158) gives

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle^2 - \left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right) \left\langle \Delta\widehat{J}_x^2 \right\rangle + \frac{1}{4}\xi \left\langle \widehat{J}_z \right\rangle^2 \leq 0 \tag{333}$$

The left side is a parabolic function of $\left\langle \Delta\widehat{J}_x^2 \right\rangle$ and for this to be negative requires $\left\langle \Delta\widehat{J}_x^2 \right\rangle$ to lie between the two roots of this function, giving

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle \geq \frac{1}{2}\left\{ \left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right) - \sqrt{\left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{J}_z \right\rangle^2} \right\} \tag{334}$$

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle \leq \frac{1}{2}\left\{ \left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right) + \sqrt{\left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{J}_z \right\rangle^2} \right\} \tag{335}$$

which are the required inequalities in Eq. (159) and (160).

16 Appendix F - "Separable but Non-Local" States

4.

It is instructive to apply the various entanglement tests to the so-called separable but non-local states considered in Refs. [3], [43], for which the sub-system states are definitely *not* SSR compliant. These states should not pass the the Hillery tests [23], [33] for SSR neglected entanglement, but they may pass the entanglement tests in this paper and in Ref. [1] since these states would be regarded as SSR compliant entangled. Note that these states are consistent with the global particle number SSR, so there is no dispute about whether they are possible two mode quantum states. The issue is rather whether they should be categorised as separable or entangled, and that depends on how separable (and hence entangled) states are first defined. As discussed previously, the interferometric measurements discussed here do not enable us to choose one definition over the other - that is an issue involved swhat types of quantum states would be allowed in the separate sub-systems.

The first example of such states is the *mixture* of *two mode coherent states* is represented by the two mode density operator

$$\begin{aligned}\widehat{\rho} &= \int \frac{d\theta}{2\pi}\, |\alpha,\alpha\rangle\,\langle\alpha,\alpha| \\ &= \int \frac{d\theta}{2\pi}\, \left(|\alpha\rangle\,\langle\alpha|\right)_a \otimes \left(|\alpha\rangle\,\langle\alpha|\right)_b \end{aligned} \tag{336}$$

where $|\alpha\rangle_C$ is a one mode coherent state for mode $c = a, b$ with $\alpha = |\alpha|\, \exp(-i\theta)$, and modes a, b are associated with bosonic annihilation operators $\widehat{a}$, $\widehat{b}$. The magnitude $|\alpha|$ is fixed. This state gobally but not locally SSR compliant.

Now

$$\begin{aligned}\left\langle \widehat{a}^{\dagger}\widehat{b} \right\rangle &= Tr \int \frac{d\theta}{2\pi}\, \widehat{a}^{\dagger}\widehat{b}\, \left(|\alpha\rangle\,\langle\alpha|\right)_a \otimes \left(|\alpha\rangle\,\langle\alpha|\right)_b \\ &= Tr \int \frac{d\theta}{2\pi}\, \left(|\alpha\rangle\,\langle\alpha|\,\widehat{a}^{\dagger}\right)_a \otimes \left(\widehat{b}\,|\alpha\rangle\,\langle\alpha|\right)_b \\ &= |\alpha|^2 \end{aligned} \tag{337}$$

But

$$\begin{aligned}\left\langle \widehat{a}^{\dagger}\widehat{a}\,\widehat{b}^{\dagger}\widehat{b} \right\rangle &= Tr \int \frac{d\theta}{2\pi}\, \left(\widehat{a}^{\dagger}\widehat{a}\,|\alpha\rangle\,\langle\alpha|\right)_a \otimes \left(\widehat{b}^{\dagger}\widehat{b}\,|\alpha\rangle\,\langle\alpha|\right)_b \\ &= \int \frac{d\theta}{2\pi}\, \left(\langle\alpha|\,\widehat{a}^{\dagger}\widehat{a}\,|\alpha\rangle\right)_a \otimes \left(\langle\alpha|\,\widehat{b}^{\dagger}\widehat{b}\,|\alpha\rangle\right)_b \\ &= |\alpha|^4 \end{aligned} \tag{338}$$

Hence we have $|\left\langle \widehat{a}^{\dagger}\widehat{b} \right\rangle|^2 > 0$ and $|\left\langle \widehat{a}^{\dagger}\widehat{b} \right\rangle|^2 = \left\langle \widehat{a}^{\dagger}\widehat{a}\,\widehat{b}^{\dagger}\widehat{b} \right\rangle$. This shows the state is SSR compliant entangled. However it fails the Hillery test for SSR neglected entanglement which is consistent with being a SSR neglected separable state from the [3], [43] viewpoint.

The second example of such states has an overall density operator which is a statistical mixture given by

$$\begin{aligned}
\widehat{\rho} &= \frac{1}{4}(|\psi_1\rangle\langle\psi_1|)_a \otimes |\psi_1\rangle\langle\psi_1|)_b + \frac{1}{4}(|\psi_i\rangle\langle\psi_i|)_a \otimes |\psi_i\rangle\langle\psi_i|)_b \\
&\quad + \frac{1}{4}(|\psi_{-1}\rangle\langle\psi_{-1}|)_a \otimes |\psi_{-1}\rangle\langle\psi_{-1}|)_b + \frac{1}{4}(|\psi_{-i}\rangle\langle\psi_{-i}|)_a \otimes |\psi_{-i}\rangle\langle\psi_{-i}|)_b
\end{aligned} \tag{339}$$

where $|\psi_\omega\rangle = (|0\rangle + \omega|1\rangle)/\sqrt{2}$, with $\omega = 1, i, -, -i$. The $|\psi_\omega\rangle$ are superpositions of zero and one boson states and consequently the local particle number SSR is violated by each of the sub-system density operators $|\psi_\omega\rangle\langle\psi_\omega|)_a$ and $|\psi_\omega\rangle\langle\psi_\omega|)_b$.

Now using $\widehat{b}|\psi_\omega\rangle = (\omega|0\rangle)/\sqrt{2}$, $\langle\psi_\omega|\widehat{a}^\dagger = (\langle 0|\omega^*)/\sqrt{2}$ and $|\omega|^2 = 1$

$$\begin{aligned}
\left\langle \widehat{a}^\dagger\widehat{b} \right\rangle &= Tr\frac{1}{4}\sum_\omega (\widehat{a}^\dagger|\psi_\omega\rangle\langle\psi_\omega|_a) \otimes (\widehat{b}|\psi_\omega\rangle\langle\psi_\omega|_b) \\
&= \frac{1}{4}\sum_\omega \langle\psi_\omega|\widehat{a}^\dagger|\psi_\omega\rangle_a \langle\psi_\omega|\widehat{b}|\psi_\omega\rangle_b \\
&= \frac{1}{4}\sum_\omega \frac{1}{2}\omega^*\frac{1}{2}\omega \\
&= \frac{1}{4}
\end{aligned} \tag{340}$$

But

$$\begin{aligned}
\left\langle \widehat{a}^\dagger\widehat{a}\widehat{b}^\dagger\widehat{b} \right\rangle &= Tr\frac{1}{4}\sum_\omega (\widehat{a}^\dagger\widehat{a}|\psi_\omega\rangle\langle\psi_\omega|_a) \otimes (\widehat{b}^\dagger\widehat{b}|\psi_\omega\rangle\langle\psi_\omega|_b) \\
&= \frac{1}{4}\sum_\omega \langle\psi_\omega|\widehat{a}^\dagger\widehat{a}|\psi_\omega\rangle_a \langle\psi_\omega|\widehat{b}^\dagger\widehat{b}|\psi_\omega\rangle_b \\
&= \frac{1}{4}\sum_\omega \frac{1}{2}|\omega|^2\frac{1}{2}|\omega|^2 \\
&= \frac{1}{4}
\end{aligned} \tag{341}$$

Hence we have $|\left\langle \widehat{a}^\dagger\widehat{b} \right\rangle|^2 > 0$ and $|\left\langle \widehat{a}^\dagger\widehat{b} \right\rangle|^2 < \left\langle \widehat{a}^\dagger\widehat{a}\widehat{b}^\dagger\widehat{b} \right\rangle$. This shows the state is SSR compliant entangled. However it fails the Hillery test for entanglement, so is consistent with being a SSR neglected separable state [3], [43] viewpoint. It should be noted however that the density operator can also be written as

$$\begin{aligned}
\widehat{\rho} &= \frac{1}{4}(|0\rangle\langle 0|)_A \otimes |0\rangle\langle 0|)_B + \frac{1}{4}(|1\rangle\langle 1|)_A \otimes |1\rangle\langle 1|)_B \\
&\quad + \frac{1}{2}(|\Psi_+\rangle\langle\Psi_+|)_{AB}
\end{aligned} \tag{342}$$

where $|\Psi_+\rangle_{AB} = (|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)/\sqrt{2}$. In this form the terms correspond to a statistical mixture of states with $0, 1, 2$ bosons. The first two terms correspond to separable states, in which the sub-system density operators are SSR

compliant. The final term however is a one boson Bell state which is generally regarded as the paradigm of a two mode entangled state. Hence regarding the overall state as separable is highly questionable.

17 Appendix G - Derivation of Interferometer Results

17.1 General Theory - Two Mode Interferometer

Introducing the free and interaction evolution operators via

$$\begin{aligned}\widehat{U} &= \widehat{U}_0\,\widehat{U}_{int} \\ \widehat{U}_0 &= \exp(-i\widehat{H}_0 t/\hbar)\end{aligned} \tag{343}$$

it is straightforward to show that for

$$\widehat{M} = \frac{1}{2}(\widehat{b}^{\dagger}\widehat{b} - \widehat{a}^{\dagger}\widehat{a}) \tag{344}$$

we have

$$\begin{aligned}\left\langle \widehat{M} \right\rangle &= Tr(\widehat{M}_{\,H}\,\widehat{\rho}) \\ \left\langle \Delta\widehat{M}^2 \right\rangle &= Tr(\left\{ \widehat{M}_H - \left\langle \widehat{M}_H \right\rangle \right\}^2\,\widehat{\rho})\end{aligned} \tag{345}$$

giving the mean and variance in terms of the input density operator and interaction picture Heisenberg operators

$$\begin{aligned}\widehat{M}_H &= \frac{1}{2}(\widehat{b}_H^{\dagger}\widehat{b}_H - \widehat{a}_H^{\dagger}\widehat{a}_H) \\ \widehat{b}_H &= \widehat{U}_{int}^{-1}\,\widehat{b}\,\widehat{U}_{int} \qquad \widehat{a}_H = \widehat{U}_{int}^{-1}\,\widehat{a}\,\widehat{U}_{int}\end{aligned} \tag{346}$$

where we have used the results $\widehat{U}_0^{-1}\,\widehat{b}\,\widehat{U}_0 = \exp(-i\omega_b t)\,\widehat{b}$ and $\widehat{U}_0^{-1}\,\widehat{a}\,\widehat{U}_0 = \exp(-i\omega_a t)\,\widehat{a}$.

The interaction picture Heisenberg operators satisfy

$$i\hbar\frac{\partial}{\partial t}\widehat{b}_H = [\widehat{b}_H, \widehat{V}_H] \qquad i\hbar\frac{\partial}{\partial t}\widehat{a}_H = [\widehat{a}_H, \widehat{V}_H] \tag{347}$$

where

$$\begin{aligned}\widehat{V}_H &= \mathcal{A}(t)\exp(-i\omega_0 t)\,\exp(i\phi)\,\widehat{b}_H^{\dagger}\widehat{a}_H\,\exp(+i\omega_{ba}t) \\ &\quad +\mathcal{A}(t)\exp(+i\omega_0 t)\,\exp(-i\phi)\,\widehat{a}_H^{\dagger}\widehat{b}_H\,\exp(-i\omega_{ba}t) \\ &= \mathcal{A}(t)\,\exp(i\phi)\,\widehat{b}_H^{\dagger}\widehat{a}_H + \mathcal{A}(t)\,\exp(-i\phi)\,\widehat{a}_H^{\dagger}\widehat{b}_H\end{aligned} \tag{348}$$

for resonance.

We then find that the Heisenberg picture operators satisfy coupled linear equations

$$i\hbar\frac{\partial}{\partial t}\widehat{b}_H = \mathcal{A}(t)\,\exp(+i\phi)\,\widehat{a}_H \qquad i\hbar\frac{\partial}{\partial t}\widehat{a}_H = \mathcal{A}(t)\,\exp(-i\phi)\,\widehat{b}_H \tag{349}$$

which after replacing the time t by the area variable s then involve time independent coefficients

$$i\frac{\partial}{\partial s}\widehat{b}_H(s) = \exp(+i\phi)\,\widehat{a}_H(s) \qquad i\frac{\partial}{\partial s}\widehat{a}_H(s) = \exp(-i\phi)\,\widehat{b}_H(s) \tag{350}$$

The equations can then be solved via Laplace transforms giving

$$\widehat{b}_H(s,\phi) = \cos s\,\widehat{b} - i\exp(i\phi)\,\sin s\,\widehat{a} \qquad \widehat{a}_H(s,\phi) = -i\exp(-i\phi)\,\sin s\,\widehat{b} + \cos s\,\widehat{a} \tag{351}$$

where now $2s$ is the area for the classical pulse.

Hence we have in general

$$\begin{aligned} \widehat{M}_H(2s,\phi) &= \frac{1}{2}(\widehat{b}_H^\dagger(s,\phi)\widehat{b}_H(s,\phi) - \widehat{a}_H^\dagger(s,\phi)\widehat{a}_H(s,\phi)) \\ &= \sin 2s\,(\sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y) + \cos 2s\,\widehat{S}_z \end{aligned} \tag{352}$$

The versatility of the measurement follows from the range of possible choices of the pulse area $2s$ and the phase ϕ.

Writing $2s = \theta$ we can then substitute into Eq.(345) to obtain results for $\left\langle \widehat{M} \right\rangle$ and $\left\langle \Delta\widehat{M}^2 \right\rangle$. These are set out in SubSection 7.3 in Eqs. (218) and (219) in terms of the mean values of the spin operators and the matrix elements of the covariance matrix for the spin operators. all for the quantum state$\widehat{\rho}$.

17.2 Beam Splitter and Phase Changer

For the *beam splitter* we have $2s = \pi/2$ and ϕ (variable) so that

$$\widehat{M}_H(\frac{\pi}{2},\phi) = \sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y \tag{353}$$

whilst for the *phase changer* we have $2s = \pi$ and ϕ (arbitrary) so that

$$\widehat{M}_H(\pi,\phi) = -\widehat{S}_z \tag{354}$$

17.3 Other Measureables

We can also consider other choices for the measureable, which then enable us to determine other moments of the spin operators. A case of particular interest is the square of the population difference

$$\widehat{M_2} = \left(\frac{1}{2}(\widehat{b}^\dagger\widehat{b} - \widehat{a}^\dagger\widehat{a})\right)^2 \tag{355}$$

It is then straightforward to show for the beam splitter case with $2s = \pi/2$ and ϕ (variable)

$$\begin{aligned} \widehat{M}_{2H}(\frac{\pi}{2},\phi) &= \left(\sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y\right)^2 \\ &= \sin^2\phi\,(\widehat{S}_x)^2 + \cos^2\phi\,(\widehat{S}_y)^2 + \sin\phi\,\cos\phi\,(\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x) \end{aligned} \tag{356}$$

Hence

$$\left\langle \widehat{M_2} \right\rangle = \sin^2\phi\,\left\langle (\widehat{S}_x)^2 \right\rangle + \cos^2\phi\,\left\langle (\widehat{S}_y)^2 \right\rangle + \sin\phi\,\cos\phi\,\left\langle (\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x) \right\rangle \tag{357}$$

showing that the mean for the new observable $\widehat{M_2}$ is a sinusoidal function of the BS interferometer variable ϕ with coefficients that depend on the means of $\widehat{S}_x^2$, $\widehat{S}_y^2$ and $\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x$.

17.4 General Theory - Multi-Mode Interferometer

The derivation follows the same steps as in SubSection 17.1. However here we have the results $\widehat{U}_0^{-1}\,\widehat{b}_i\,\widehat{U}_0 = \exp(-i(\omega_b + \omega_i)t)\,\widehat{b}_i$ and $\widehat{U}_0^{-1}\,\widehat{a}_i\,\widehat{U}_0 = \exp(-i(\omega_a + \omega_i)t)\,\widehat{a}_i$. The factors involving $\exp(-i\omega_i)t$ cancel out in the derivation of the Heisenberg equations, which here are

$$i\hbar\frac{\partial}{\partial t}\widehat{b}_{iH} = \mathcal{A}(t)\,\exp(+i\phi)\,\widehat{a}_{iH} \qquad i\hbar\frac{\partial}{\partial t}\widehat{a}_{iH} = \mathcal{A}(t)\,\exp(-i\phi)\,\widehat{b}_{iH} \tag{358}$$

and the solutions are

$$\widehat{b}_{iH}(s,\phi) = \cos s\,\widehat{b}_i - i\exp(i\phi)\,\sin s\,\widehat{a}_i \qquad \widehat{a}_{iH}(s,\phi) = -i\exp(-i\phi)\,\sin s\,\widehat{b}_i + \cos s\,\widehat{a}_i \tag{359}$$

where now $2s$ is the area for the classical pulse.

Hence we have in general

$$\begin{aligned}\widehat{M}_H(2s,\phi) &= \frac{1}{2}\sum_i(\widehat{b}_{iH}^\dagger(s,\phi)\widehat{b}_{iH}(s,\phi) - \widehat{a}_{iH}^\dagger(s,\phi)\widehat{a}_{iH}(s,\phi)) \\ &= \sin 2s\,(\sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y) + \cos 2s\,\widehat{S}_z \end{aligned} \tag{360}$$

This leads to the same formal results (235) and (236) for the mean and variance. The versatility of the measurement again follows from the range of possible choices of the pulse area $2s$ and the phase ϕ.

18 Appendix H - Limits on Interferometry Tests

The *tests* for entanglement in a particular quantum state are given in terms of the *mean value* and *variance* for certain physical quantities. Interferometers are used to enable these means and variances to be determined from measurements on *another* physical quantity when either the state being tested is acted upon by the interferometer or it is being unaffected. Quantum theory enables us to predict two things. Firstly, for any physical quantity $\widehat{M}$ we can predict the *possible values* that measurements could result in. Results from a succession of measurements would confirm what these values are. Secondly, for any quantum state, we can predict the *probability* that measurement leads to a specific value. A single measurement only yields one of the possible values, so *independent repetitions* of such measurements is needed to confirm what the probabilities for measuring particular values are - ideally an infinite number of repeated measurements would be required. If this was possible, the computed mean $\left\langle \widehat{M} \right\rangle$ and variance $\left\langle \Delta\widehat{M}^2 \right\rangle$ of the measurements for the physical quantity $\widehat{M}$ would confirm the quantum theory predictions for any quantum state. A finite but large number of independent measurements - each based on the *same* probabiity distribution for the possible results, would enable us to *estimate* the *actual* mean and variance of the measured values from the *sample* measurements. These estimates would not be precisely accurate. The question is - how *big* would the sample of repeated measurements need to be for the purpose of using the estimated mean and variance in the tests for *entanglement* ?

Statistical theory in the form of the *central limit theorem* [44] can be applied here. This tells us if the number R of repeated measurements is large, then the mean of the *sample* measurements approaches the *true* mean and the *variance* in the *sample estimation* of the *mean* is given by the *true variance* divided by R

$$\begin{aligned} \left\langle \widehat{M} \right\rangle_{sample} &\rightarrow \left\langle \widehat{M} \right\rangle \\ \left\langle \Delta \left\langle \widehat{M} \right\rangle^2 \right\rangle_{sample} &\rightarrow \frac{\left\langle \Delta\widehat{M}^2 \right\rangle}{R} \end{aligned} \tag{361}$$

We can use our theoretical estimate of the variance $\left\langle \Delta\widehat{M}^2 \right\rangle$ to get an idea of how large the sample of measurements must be in order that the standard deviation of the sample estimate for the mean is small enough that the mean can confidently be stated to exceed or be less than the quantity on the other side of the inequality in the entanglement test.

19 Appendix I - Relative Phase State

The *relative phase eigenstate* (see [6], [47]) for an N boson two mode system has provided an important example of different outcomes for the simple spin squeezing and Hillery spin squeezing tests, so here its properties are set out in more detail. The results for interferometric measurements on the relative phase state are also presented.

The relative phase state is a globally compliant entangled state of the sub-systems a and b and is defined by

$$|N,\theta_p\rangle = \frac{1}{\sqrt{N+1}}\sum_{k=-N/2}^{N/2}\exp(ik\theta_p^N)\,|N/2-k\rangle_a\,|N/2+k\rangle_b \tag{362}$$

where $\theta_p^N = p(2\pi/(N+1))$, $p=-N/2,-N/2+1,..,+N/2$ is a quasi-continuum of $N+1$ equispaced phase eigenvalues, and $|N/2-k\rangle_a$, $|N/2+k\rangle_b$ are Fock states for sub-systems a and b. The Hermitian relative phase operator $\widehat{\Theta}_N$ for N boson states is then defined as

$$\widehat{\Theta}_N = \sum_p \theta_p^N\ |N,\theta_p\rangle\langle N,\theta_p| \tag{363}$$

and $|N,\theta_p\rangle$ is an eigenvector with eigenvalue θ_p^N.

Since these states are entangled with maximum *mode entropy*, are *spin squeezed* and are *fragmented* BEC (two modes have macroscopic occupancy) it is of some interest to examine their interferometric properties for the simple beam splitter interfometer. As shown in [6] the relative phase state has the following mean values for the spin operators when $\widehat{\rho} = |N,\theta_p\rangle\langle N,\theta_p|$

$$\left\langle \widehat{S}_x\right\rangle_\rho = \frac{N\pi}{8}\cos\theta_p \qquad \left\langle \widehat{S}_y\right\rangle_\rho = -\frac{N\pi}{8}\sin\theta_p \qquad \left\langle \widehat{S}_z\right\rangle_\rho = 0 \tag{364}$$

so that for the measurable

$$\left\langle \widehat{M}\right\rangle = \frac{N\pi}{8}\sin(\phi-\theta_p) \tag{365}$$

We thus have a large amplitude - proportional to N - sinusoidal dependence for the mean value of the measureable on the interferometer phase detuning $(\phi-\theta_p)$, and which goes to zero when $\phi=\theta_p$. Since we never have both $\left\langle \widehat{S}_x\right\rangle_\rho$ and $\left\langle \widehat{S}_y\right\rangle_\rho$ equal to zero the simple correlation test confirms that the relative phase eigenstate is entangled.

As mentioned above, the relative phase state is highly spin squeezed. To describe this it is convenient to introduce rotated spin operators $\widehat{J}_x$, $\widehat{J}_y$ and $\widehat{J}_z$ given by (see Ref [6], Eqn. 179)

$$\begin{aligned}\widehat{J}_x &= \widehat{S}_z\\ \widehat{J}_y &= \sin\theta_p\,\widehat{S}_x + \cos\theta_p\,\widehat{S}_y\\ \widehat{J}_z &= -\cos\theta_p\,\widehat{S}_x + \sin\theta_p\,\widehat{S}_y\end{aligned} \tag{366}$$

The new spin operators are Schwinger spin operators for *new modes* c, d where

$$\widehat{a} = -\exp(\frac{1}{2}i\theta_p)\left(\widehat{c} - \widehat{d}\right)/\sqrt{2} \quad \widehat{b} = -\exp(-\frac{1}{2}i\theta_p)\left(\widehat{c} + \widehat{d}\right)/\sqrt{2} \tag{367}$$

and the relative phase state also an *entangled* state for *new* modes. This can be shown by substituting for the $|N/2-k\rangle_a$ and $|N/2+k\rangle_b$ in terms of Fock states for the new modes c, d.

These new angular momentum operators are *principal spin operators* for which the covariance matrix is diagonal. For the mean values

$$\left\langle \widehat{J}_x \right\rangle_\rho = 0 \qquad \left\langle \widehat{J}_y \right\rangle_\rho = 0 \qquad \left\langle \widehat{J}_z \right\rangle_\rho = -\frac{N\pi}{8} \tag{368}$$

In terms of spin operators discussed above (see Eqs. (238) and (239)) we have $\widehat{J}_x = \widehat{S}_z$, $\widehat{J}_y = \widehat{S}_x^{\#}(\frac{3\pi}{2}+\theta_p)$ and $\widehat{J}_z = \widehat{S}_y^{\#}(\frac{3\pi}{2}+\theta_p)$ so the variances for $\widehat{J}_y$ and $\widehat{J}_z$ can be measured using the simple BS interferometer, and the mean for $\widehat{J}_x$ is also measureable by simply measuring the mean population difference without subjecting the relative phase eigenstate to the BS interaction.

Inverting these expressions and substituting gives the measureable in terms of the new spin operators

$$\widehat{M}_H = \cos(\phi-\theta_p)\,\widehat{J}_y - \sin(\phi-\theta_p)\,\widehat{J}_z \tag{369}$$

Hence we find for the variance of the measureable

$$\begin{aligned}\left\langle \Delta\widehat{M}^2 \right\rangle &= \cos^2(\phi-\theta_p)\,C(\widehat{J}_y,\widehat{J}_y) + \sin^2(\phi-\theta_p)\,C(\widehat{J}_z,\widehat{J}_z) \\ &\quad -2\sin(\phi-\theta_p)\,\cos(\phi-\theta_p)\,C(\widehat{J}_y,\widehat{J}_z) \end{aligned}\tag{370}$$

As $\widehat{J}_x$,$\widehat{J}_y$ and $\widehat{J}_z$ are principal spin operators $C(\widehat{J}_y,\widehat{J}_z) = 0$ and substituting for the variances $C(\widehat{J}_y,\widehat{J}_y) = 1/4 + 1/8\,\ln N$ and $C(\widehat{J}_z,\widehat{J}_z) = (1/6-\pi^2/64)\,N^2$ (see [6]) we get for the variance of the measureable for an input relative phase eigenstate

$$\begin{aligned}\left\langle \Delta\widehat{M}^2 \right\rangle &= \cos^2(\phi-\theta_p)\,(\frac{1}{4}+\frac{1}{8}\ln N) + \sin^2(\phi-\theta_p)\,(\frac{1}{6}-\frac{\pi^2}{64})\,N^2 \\ &\approx \frac{1}{4} + (\phi-\theta_p)^2\,(\frac{1}{6}-\frac{\pi^2}{64})\,N^2 \end{aligned}\tag{371}$$

for $\phi \approx.\ \theta_p$. The other variance is $C(\widehat{J}_x,\widehat{J}_x) = (1/12)\,N^2$. The variance for the measurable depends sinusoidally on $2(\phi-\theta_p)$. Thus the quantum noise in the measureable also goes to essentially zero at $\phi = \theta_p$, when the mean value $\left\langle \widehat{M} \right\rangle$ also goes to zero. The width $\Delta\phi$ for this low noise window scales as $1/N$ - which corresponds to the Heisenberg limit. At the zero of the mean value, the relative fluctuation varies as $1/N$ as in the Heisenberg limit. Since for $\phi = \theta_p$ we have $\widehat{M}_H = \widehat{J}_y = \widehat{S}_x^{\#}(\frac{3\pi}{2}+\theta_p)$ and $\left\langle \Delta\widehat{M}^2 \right\rangle = (\frac{1}{4}+\frac{1}{8}\ln N)$

whilst $\left\langle \widehat{S}_z \right\rangle_\rho = \left\langle \widehat{J}_z \right\rangle_\rho = -\frac{N\pi}{8}$. Thus the spin squeezing test in Eq.(241) is satisfied, confirming again that the relative phase eigenstate is an entangled state of modes a and b.

In regard to *particle entanglement* [45], [46] with $\widehat{\rho} = |N, \theta_p\rangle \langle N, \theta_p|$ and with $n_a = (N/2 - k)$, $n_b = (N/2 + k)$, the quantities in Eqs. (**132**) and (**133**) **of paper 1** are given by

$$\begin{aligned}
\widehat{\rho}^{(n_a n_b)} &= \frac{1}{N+1} |N/2 - k\rangle_a \langle N/2 - k|_a \otimes |N/2 + k\rangle_b \langle N/2 + k|_b \quad (372) \\
P_{n_a n_b} &= \frac{1}{N+1} \quad (373)
\end{aligned}$$

and since $\widehat{\rho}^{(n_a n_b)}$ is a separable state, it follows that $E_P(\ \widehat{\rho}) = 0$. Thus the measure of particle entanglement is zero for what is clearly an *entangled* state. Hence the particle entanglement measure has not detected entanglement in this example.

The relative phase state is therefore a promising candidate for use as an input state in two mode interferometry. More elaborate interferometers where the interferometric variable is associated with other systems whose parameters are to be measured might be developed. The main issue would be whether such a relative phase state could be prepared. This is an issue being dealt with elsewhere [47].